Low-Energy Theory of Disordered Graphene

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At low values of external doping, graphene displays a wealth of unconventional transport properties. Perhaps most strikingly, it supports a robust "metallic" regime, with universal conductance of the order of the conductance quantum. We here apply a combination of mean-field and bosonization methods to explore the large scale transport properties of the system. We find that, irrespective of the doping level, disordered graphene is subject to the common mechanisms of Anderson localization. However, at low doping a number of renormalization mechanisms conspire to protect the conductivity of the system, to an extend that strong localization may not be seen even at temperatures much smaller than those underlying present experimental work.

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The low-energy physics of two-dimensional carbon, or graphene [1,2], is governed by the presence of two generations of Dirac fermions. A conspiracy of these quasiparticles with various channels of impurity scattering leads to a wealth of intriguing transport phenomena which have attracted a lot of recent attention (see Refs. [3,4] for a list of early references). Prominent examples of the unorthodox conduction properties of graphene monolayers include a quantum Hall effect with unconventional plateaux quantization $\sigma_{xy} = (2n+1)e^2/h$, and the experimental observation of a minimal universal conductivity $\sigma_{xx} \sim e^2/h$ at low concentration of dopands, $E_F \sim \tau^{-1}$ [1,2] (τ^{-1} is the elastic scattering rate and E_F the Fermi energy, vanishing in the limit of zero external doping). While many of graphene's transport properties have been successfully addressed theoretically, a number of important questions remain open. Specifically, the nature of the low-energy regime $E_F \lesssim \tau^{-1}$—where the deviations from the behavior of conventional metals are most pronounced—is not fully understood. Relying on calculations to lowest order in perturbation theory in the disorder concentration [5], the experimental results have been interpreted [1] in terms of some delocalized, genuinely metallic phase at the band center. However, in the very low-energy regime, where the conductivity begins to level off to a disorder independent value, perturbation theory breaks down and does not really help to elucidate the physics of the system.

Building on a combination of mean-field analysis and bosonized perturbation theory around the clean Dirac fermion fixed point, $E_F = 0$, we here propose an effective low-energy field theory of system. Reflecting the large scale symmetries of the problem [6,7], the relevant field theory at strong doping ($E_F\tau \gg 1$) is a standard nonlinear σ model of either orthogonal or unitary symmetry (depending on whether time reversal invariance is broken). The bare coupling constant of the theory [cf. Eq. (3) below] $\sigma_{xx} \sim E_F\tau \gg 1$ is large, implying that localization corrections to the Drude conductivity are comparatively weak. Our main finding is that the weak doping regime, $E_F\tau \ll 1$, is also described by the nonlinear σ model. This result implies that at all doping levels disordered graphene is subject to common mechanisms of Anderson localization. Specifically, the smallness of the bare coupling constant controlling the theory at $E_F\tau < 1$—loosely to be identified with the disorder independent "minimal universal conductivity" $\sigma_{xx} = \mathcal{O}(1)$ of the system—signals that localization and fluctuation phenomena will be especially pronounced at small doping. Below we will discuss a number of mechanisms which may explain why these fluctuations have not yet been observed experimentally. We conclude by briefly considering the quantum Hall physics of the system.

In the absence of disorder, graphene's two generations of low-energy Dirac fermions are described by the Hamiltonian [4] $\hat{H}_{\rm eff} = -iv\partial_x\sigma_x \otimes \tau_3 - iv\partial_x\sigma_y$, where v is the Fermi velocity, the Pauli matrices σ_i connect between the two sublattices A and B of graphene's hexagonal host lattice, and the Pauli matrices τ_i act in two-component nodal space. The off diagonality of the Hamiltonian in sublattice space implies chirality, $[\hat{H}, \sigma_3]_+ = 0$.

In the presence of all possible channels of disorder scattering, $\hat{H}$ generalizes to (outer block structure in nodal space, 2×2 subblocks in sublattice space),

$$\hat{H} = \begin{bmatrix} V_A & -iv\partial + V & W_A & W^{+-} \\ -iv\bar{\partial} + \bar{V} & V_B & W^{-+} & W_B \\ \bar{W}_A & \bar{W}^{-+} & V_A & iv\bar{\partial} + V' \\ \bar{W}^{+-} & \bar{W}_B & iv\partial + \bar{V}' & V_B \end{bmatrix}, \tag{1}$$

where $\partial = \partial_x + i\partial_y$, and V_C and W_C, respectively, describe the soft (intranode) and the hard (internode) scattering due to defect sites, next nearest neighbor hopping, or disorder in adjacent substrates. Diagonal in sublattice space, these scattering amplitudes violate the chiral symmetry of the clean system. Similarly, the chiral symmetry preserving matrix elements V' and W^{+-}, W^{-+} represent the intra- and internode amplitudes, respectively, of defective bond hopping.

Referring to Ref. [7] for a much more comprehensive discussion, we briefly comment on the *symmetries of the problem*: (i) If all hopping matrix elements of the original problem are real (complex), the low-energy Hamiltonian does (not) obey the symmetry relation $\hat{H} = \tau_1 \hat{H}^T \tau_1$, thus falling into the Wigner-Dyson orthogonal (unitary) symmetry class. However, (ii) depending on the microscopic realization of the disorder, some of the amplitudes entering the bare Hamiltonian may be much smaller (even negligibly small at the length scales probed in current experiment) than others. If this happens, the approximate realization of symmetries beyond Wigner-Dyson orthogonal or unitary may result in various transport anomalies, including weak antilocalization phenomena at intermediate length scales [7–9]. However, not protected by rigid exclusion principles, these structures are transient in nature and will not affect the physics at the largest distance scales; we will therefore assume the presence of all disorder channels throughout.

Transport properties of the system are described by disorder averaged products of retarded and advanced Green functions $\langle (E_F - \hat{H} + i0)^{-1}_{\mathbf{x}_1,\mathbf{x}_2} (E_F - \hat{H} - i0)^{-1}_{\mathbf{x}_3,\mathbf{x}_4} \cdots \rangle_{\text{dis}}$. To compute such correlation functions we consider the replica coherent state field integral (generalization to a supersymmetry formulation is straightforward):

$$\mathcal{Z} = \int D(\bar{\psi}, \psi) \left\langle \exp\left(i \int d^2x \bar{\psi}(i\delta\sigma_3^{\text{ar}} + E_F - \hat{H})\psi \right) \right\rangle_{\text{dis}}, \tag{2}$$

where $\psi = \{\psi_{s,a,\nu}(\mathbf{x})\}$ is a $(2 \times R \times 4)$-component Grassmann field, $s = 1, 2$ is a two-component index distinguishing between retarded and advanced indices, $a = 1, \ldots, R$ is a replica index, $\nu = 1, \ldots, 4$ enumerates sublattice and nodal sectors, and $\sigma_3^{\text{ar}} \equiv \delta_{ss'}(-)^{s+1}$ is a Pauli matrix causing infinitesimal symmetry breaking in advanced-retarded space.

Beginning with the *strong doping regime*, $E_F\tau \gg 1$, we next outline the construction of the low-energy field theory describing the large scale behavior of the system (i.e., at length scales larger than the elastic mean free path, l). The construction of the theory at large doping follows standard procedures (cf. Refs. [10,11] for review), and we restrict ourselves to a brief recapitulation of the central construction steps: the action in (2) is invariant under uniform unitary rotations $\psi \to U\psi$, where $U \in U(2R)$. Averaging over all scattering channels in (1) generates a mean-field fermion self-energy $i\delta\sigma_3^{\text{ar}} \to i\Sigma\sigma_3^{\text{ar}}$, breaking the symmetry down to $U(R) \times U(R)$. Along with this spontaneous symmetry breaking, Goldstone modes $Q \in U(2R)/U(R) \times U(R)$ appear. The effective action $S[Q]$ controlling the low-energy partition function $\mathcal{Z}_{\text{eff}} \equiv \int DQ \exp(-S[Q])$ of these modes derives from the (unique) parity invariant second order derivative operator, $-\int d^2x \,\text{Tr}(\partial_\mu Q \partial_\mu Q)$. Gauge arguments (but also the microscopic construction) state that the bare coupling of this operator is $(1/8)\times$ (the dimensionless Drude longitudinal conductivity, σ_{xx}), i.e.,

$$S[Q] = -\frac{\sigma_{xx}}{8} \int d^2x \,\text{Tr}(\partial_\mu Q \partial_\mu Q), \tag{3}$$

which is the standard nonlinear σ-model action of time reversal invariance broken weakly disordered metallic systems. [In the time reversal invariant case, $Q \in \text{Sp}(4R)/\text{Sp}(2R) \times \text{Sp}(2R)$ and $\sigma_{xx}/8 \to \sigma_{xx}/16$.] Specifically, graphene's Drude conductivity is given by [5] $\sigma_{xx} = 2\tau E_F E_F \tau \gg 1$. Referring for an in depth discussion of the localization properties of the theory (3) to Ref. [11], we note that at intermediate length scales $L > l$ this bare value will be modified by weak localization corrections (for a comprehensive discussion of weak localization effects in graphene at $E_F\tau > 1$ and different channels of disorder scattering, see Refs. [6,7]), before strong Anderson localization sets in at exponentially large scales $L \sim \xi \equiv l \exp(\text{const} \times \sigma_{xx}^2)$ (unitary symmetry) or $\xi \equiv l \exp(\text{const} \times \sigma_{xx})$ (orthogonal symmetry). However, before discussing the phenomenology of the system any further, let us explore what happens in the strong disorder, or *weak doping regime*, $E_F\tau < 1$.

While the mean-field construction of (3) essentially relied on the presumed largeness of the parameter $E_F\tau$, changes in the bare disorder concentration are not expected to change the basic symmetries of the system. One may thus suspect that the low-energy theory retains its integrity as we push on into the strong disorder regime $E_F\tau < 1$. In the following we will apply bosonization techniques (similar to those previously used in the context of the d-wave superconductor [12,13]) to confirm that this conjecture is correct. Readers not interested in the technicalities of this construction may directly advance to the discussion two paragraphs further down.

Our starting point is Witten's seminal result [14] according to which a system of N free Dirac fermions may be equivalently represented in terms of a level 1 Wess-Zumino-Witten (WZW) model with target manifold $U(N)$: $\int d^2x(\bar{\psi}_+ \partial \psi_+ + \bar{\psi}_- \bar{\partial} \psi_-) \leftrightarrow S[M]$, where $\psi_\pm$ are N-component Grassmann fields and $M \in U(2R)$ is an $N \times N$ unitary matrix field with action $S[M] = S_0[M] + \frac{i}{12\pi}\Gamma[M]$. Here, $S_0[M] = \frac{1}{8\pi}\int d^2x \,\text{Tr}(\partial_\mu M \partial_\mu M^{-1})$ and $\Gamma[M]$ is the topological WZW action whose detailed structure will not be of relevance throughout [15]. Specifically, the replica theory of clean graphene with its two generations (K and K') of $2R$ Dirac fermions is represented in terms of two matrix fields $M, M' \in U(2R)$, with decoupled action $S[M, M'] = S[M] + S'[M']$. Here $S[M] = S_0[M] + \frac{i}{12\pi}\Gamma[M]$ and $S'[M'] = S_0[M'] - \frac{i}{12\pi}\Gamma[M']$ where the relative sign in front of the WZW actions signals the different orientation (or parity) of the two low-energy islands. The role of disorder scattering may now be explored by averaging the exponentiated impurity matrix elements in the prototypical action (2) over the disorder distribution functions (here assumed to be Gaussian correlated). This results in the appearance of four-fermion operators, which may subsequently be represented in bosonic language [14]. Exemplifying this procedure on the internode chiral symmetry preserving scattering amplitudes, we find

$$\langle e^{-i\int d^2x[\bar\psi_+ W^{+-}\psi'_+ + \bar\psi'_- \bar W^{+-}\psi_-]}\rangle_{\rm dis} = e^{\gamma_{\rm AB}^{KK'}\int d^2x\,{\rm Tr}(\psi_-\bar\psi_+\psi'_+\bar\psi'_-)} \longrightarrow e^{\gamma_{\rm AB}^{KK'}a^{-2}\int d^2x\,{\rm Tr}(MM'^{-1})},$$

where the constant $\gamma_{\rm AB}^{KK'}$ is a measure of strength of the disorder, a is the small-distance cutoff of the theory (the lattice spacing, say), and the correspondence $\psi_{-a}\bar\psi_{+b} \leftrightarrow a^{-1}M_{ab}$ was used. In a similar manner, the scattering channels V_A generates operators $-\gamma_A a^{-2}\int d^2x[{\rm Tr}(M + M'^{-1})]^{-2}$, etc.

To understand the consequences of the disorder generated modification of the theory, we note that the two operators above are examples of marginally relevant perturbations at the clean Dirac fermion fixed point, $E_F = 0$ [13,16]: For initially weak disorder, the constants γ_A and $\gamma_{\rm AB}^{KK'}$ grow to values of $\mathcal{O}(1)$ only at exponentially large length scales $l \sim a\exp({\rm const}\times v_F^2/\gamma_0)$, where γ_0 represents the bare values of the coupling. The crossover scale, l, *defines* the elastic mean free path of the theory. At larger scales, the internode scattering operator ${\rm Tr}(MM')$ enforces locking $M = M'$ of the formerly independent nodes while node and sublattice diagonal scattering, $[{\rm Tr}(M + M')]^{-1}$, implies a field reduction $M \to Q \in U(2R)/U(R)\times U(R)$ down to the manifold discussed earlier in the context of the weak coupling regime. Substituting the reduced fields into the action $S[M, M'] \to S[Q, Q]$ (for the legitimacy of naive massless field substitution within a strong coupling context, see Ref. [17]) we obtain Eq. (3) at $\sigma_{xx} = 4/\pi$. [For time reversal invariant systems, bosonization of the clean theory leads to a WZW action with target $O(8R)$. Scattering then projects onto the orthogonal variant (3).]

We conclude that the theory describing electronic transport in graphene at low doping, $E_F\tau \ll 1$, is a nonlinear σ model with disorder independent bare coupling constant $4/\pi$. However, unlike with the weak coupling situation discussed above, this description becomes valid only at length scales $l \sim a\exp({\rm const}\times v_F^2/\gamma_0)$, exponentially large in the bare inverse disorder concentration. At smaller length scales, the dynamics is essentially ballistic. (Within the framework of a large N mean-field approach, this strong disorder sensitivity of the Dirac fermion scattering mean free path, and the relevancy of the σ model at larger scales was first observed in Ref. [18].) In contrast, the mean free path $l \sim \gamma^{-1}$ at strong doping $E_F\tau \gg 1$ depends much weaker on the disorder concentration and the crossover to a multiple scattering regime occurs at smaller length scales. Second, the smallness of the coupling constant $4/\pi$ implies that the bare value of the conductivity will be subject to *strong* renormalization. More precisely, large fluctuations of the Q fields (describing the proliferation of the particle-hole modes causing Anderson localization) will lead to an exponential decay of correlation functions at length scales $\xi = \mathcal{N} l$, where $\mathcal{N}$ is a numerical factor. Accordingly, the conductivity of an infinite graphene sheet $\sigma_{xx}(T = 0) = 0$, while the conductance of a sample with extension $L \gg l$ will be exponentially small in $\exp(-{\rm const}\times L/\xi)$. However, as first pointed out in [18] on the basis of a cavalier application of a 1-loop renormalization group analysis to the strong coupling problem (3), the length scale ξ beyond which strong localization sets may still be orders of magnitudes larger than l. This observation suggests that the value $\min(L, l_\phi)$ beyond which the conductivity is strongly diminished may be much larger than the elastic mean free path, even at low values of external doping.

How can these results be reconciled with the *experimental observation* of a universal value $\sigma_{xx,{\rm expt}} \simeq 4$, significantly larger than even the bare (Drude) conductivity of the clean Dirac system? At this stage, the numerical discrepancy to the Drude value is not fully understood (see, however, Ref. [9]). As for the general robustness of the minimal conductivity, (i) elastic mean free path renormalization at small doping and (ii) the largeness of the sigma model correlation length even at strong coupling conspire to render the length scale beyond which strong localization is seen large. Finally, (iii), we repeat that the above discussion applied to a regime where all disorder scattering channels are of comparable strength. On the other hand, the experimentally observed linear dependence of the conductivity on the carrier density suggests [9] dominance of the soft (node diagonal) potential scattering, $V_A = V_B$ channel. If only this channel were operational, the relevant field theory would be two independent copies (the nodes) of sigma models of symplectic symmetry (technically, Witten's theory subject to the constraint $M = Q$, $M' = Q'$), each generating weak *antilocalization*. However, beyond a certain crossover scale, internode scattering becomes effectual, the field theory gets locked to a single copy of a theory of orthogonal symmetry and standard Anderson localization emerges; although a conspiracy of the mechanisms above may imply that the observed metallic regime is extraordinarily robust, weak localization precursors of strong localization will likely be observed at temperatures lower than the 10 K underlying present experiment.

Slightly modifying a line of arguments originally due to Pruisken [19], we finally touch upon the *quantum Hall regime*. On very general grounds, i.e., regardless of whether a magnetic fields is present, the theory contains a first order derivative operator, $S^{(1)}[T] = -\frac{\pi}{2}\times\int d^2x j_\mu(\mathbf{x})\rho(E_F)\rangle{\rm Tr}(\sigma_3^{\rm ar}T^{-1}\partial_\mu T)$, where $j_\mu(\mathbf{x})$ is the local current density operator, $\rho(E_F)$ the (disorder averaged) density of states at the Fermi energy, and $\langle \cdots \rangle_q$ the trace over single particle Hilbert space. While the bulk expectation value $\langle j_\mu \rho \rangle_q$ generally vanishes, the presence of quantum Hall edges couples $S^{(1)}$ to the theory. Substitution of the edge state system of graphene [20,21] into $S^{(1)}[T]$ obtains the boundary operator $S^{(1)}[T] = -\nu \oint ds \langle \mathrm{Tr}(\sigma_3^{\mathrm{ar}} T^{-1} \partial_s T)$, where s parametrizes the system boundary and $\nu = 2n + 1$ is the number of Landau levels below the Fermi energy. Applying Stokes theorem to convert $S^{(1)}$ to a bulk integral, adding the background action (3), and using that $\nu = \sigma_{xy}/2$ for Fermi energies intermediate between Landau levels, we obtain

$$S[Q] = -\frac{1}{8} \int d^2x \, \mathrm{Tr}[\sigma_{xx} \partial_\mu Q \partial_\mu Q - \sigma_{xy} \epsilon_{\mu\nu} Q \partial_\nu Q \partial_\mu Q]. \tag{4}$$

For high Landau levels, (4) is but a variant of Pruisken's quantum Hall action; the bare longitudinal conductivity σ_{xx} is large and the structure of graphene's edge spectrum merely enters through the odd-integer quantization of the Hall conductivity. With the lowest Landau level (LLL), $\nu = 1$, the situation is more interesting. Referring for the technicalities of bosonization in the presence of magnetic fields and system boundaries to Refs. [13,22], we note that the applicability of the action (4) to the LLL must be taken with a grain of salt: for one thing, the bosonization approach *perturbs* around the clean, nonmagnetic limit of the theory. Quantities such as the disorder averaged broadened density of states of the LLL are likely inaccessible in terms of (renormalized) perturbation theory around that point. Second, and unlike with the situation in ordinary metals, the conductivity tensor of the LLL $(\sigma_{xx}, \sigma_{xy}) = 2 \times (2/\pi, 1)$ is numerically close to the prospected value of the quantum Hall *transition* point $(\sigma_{xx}^*, \sigma_{xy}^*)$. This is important inasmuch as the quantized Hall effect transition is arguably [23,24] not described by Pruisken's theory. In practice, this may imply that strong field fluctuations (to be expected on account of the smallness of the coupling constants) readily compromise the integrity of the LLL action (4). At any rate, the identity of the proper theory of the quantum Hall transition remains as unknown as it is in normal metals.

Summarizing, we have constructed the low-energy theory of disordered graphene at large length scales. In its low doping regime, $E_F \tau < 1$, the system is described by a nonlinear σ model at strong coupling. Derived by bosonization methods, this model consistently matches the weak coupling mean-field approach, valid at large doping $E_F \tau \gg 1$. Besides the identification of the relevant low-energy theory, our main finding is that the "universal minimal conductivity" in graphene will be subject to conventional mechanisms of quantum interference at temperatures lower than those underlying current experiment. However, depending on the symmetries actually realized in the system, the temperatures below which the conductance is strongly diminished by localization may be extremely low.

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Note added.—Recently, I became aware of a paper by Aleiner and Efetov; see preceding Letter [25].

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