Local Model for Angular-Momentum Transport in Accretion Disks Driven by the Magnetorotational Instability

Martin E. Pessah,[1,2] Chi-kwan Chan,[2] and Dimitrios Psaltis[2,1]

[1]Department of Astronomy, University of Arizona, Tucson, Arizona 85721, USA

[2]Department of Physics, University of Arizona, Tucson, Arizona 85721, USA*

(Received 21 April 2006; published 1 December 2006)

We develop a local model for the exponential growth and saturation of the Reynolds and Maxwell stresses in turbulent flows driven by the magnetorotational instability. We first derive equations that describe the effects of the instability on the growth and pumping of the stresses. We highlight the relevance of a new type of correlations that couples the dynamical evolution of the Reynolds and Maxwell stresses and plays a key role in developing and sustaining the magnetorotational turbulence. We then supplement these equations with a phenomenological description of the triple correlations that lead to a saturated turbulent state. We show that the steady-state limit of the model describes successfully the correlations among stresses found in numerical simulations of shearing boxes.

DOI: 10.1103/PhysRevLett.97.221103 PACS numbers: 97.10.Gz, 95.30.Qd

Since the early days of accretion disk theory, it has been recognized that molecular viscosity cannot account for a number of observational properties of accreting objects. Shakura and Sunyaev [1] introduced a parametrization of the shear stress that has been widely used since. Much of the success of their model lies in the fact that many disk observables are determined mostly by energy balance and depend weakly on the adopted prescription [2]. However, this parametrization leaves unanswered fundamental questions on the origin of the anomalous transport and its detailed characteristics.

Strong support for the relevance of magnetic fields in accretion disks arose with the realization that differentially rotating flows with radially decreasing angular velocities are unstable when threaded by weak magnetic fields [3]. Since the discovery of this magnetorotational instability (MRI), a variety of local [4,5] and global [6–8] numerical simulations have confirmed that its long-term evolution gives rise to a sustained turbulent state and outward angular-momentum transport. However, global simulations also demonstrate that angular-momentum transfer in turbulent accretion disks cannot be adequately described by the Shakura-Sunyaev prescription. In particular, there is evidence that the turbulent stresses are not proportional to the local shear [9] and are not even determined locally [10].

Some attempts to eliminate these shortcomings have been made within the formalism of mean-field magnetohydrodynamics [11]. This has been a fruitful approach for modeling the growth of mean magnetic fields in differentially rotating media [12]. However, it has proven difficult to use a dynamo model to describe the transport of angular momentum in these systems [13,14]. This is especially true in turbulent flows driven by the MRI, where the fluctuations in the magnetic energy are larger than the mean magnetic energy, and the turbulent velocity and magnetic fields evolve simultaneously [3].

In order to overcome this difficulty, various approaches have been taken in different physical setups. Blackman and Field [15] derived a dynamical model for the nonlinear saturation of helical turbulence based on a damping closure for the electromotive force in the absence of shear. On the other hand, Kato and Yoshizawa [16,17] and Ogilvie [18] derived a set of closed dynamical equations that describe the growth and saturation of the Reynolds and Maxwell stresses in shearing flows in the absence of mean magnetic fields.

In this Letter, we relax some of the key assumptions made in previous works and develop the first local model for the dynamical evolution of the Reynolds and Maxwell tensors in turbulent magnetized accretion disks that incorporates explicitly the MRI as their source.

The equation describing the dynamical evolution of the mean angular-momentum density, $\bar{l}$, of a fluid element in an accretion disk with tangled magnetic fields is

$$\partial_t \bar{l} + \nabla \cdot (\bar{l}\, \bar{\boldsymbol{v}}) = -\nabla \cdot (r \bar{\mathcal{F}}). \tag{1}$$

Here, the overbars denote properly averaged values, $\bar{\boldsymbol{v}}$ is the mean flow velocity, and the vector $\bar{\mathcal{F}}$ characterizes the flux of angular momentum. Its radial component, $\bar{\mathcal{F}}_r \equiv \bar{R}_{r\phi} - \bar{M}_{r\phi}$, is equal to the total stress acting on a fluid element. The stresses due to correlations in the velocity field, $\bar{R}_{r\phi} \equiv \langle \rho \delta v_r \delta v_\phi \rangle$, and magnetic field fluctuations, $\bar{M}_{r\phi} \equiv \langle \delta B_r \delta B_\phi \rangle / 4\pi$, are the Reynolds and Maxwell stresses, respectively. A self-consistent accretion disk model based on the solution of the equations for the mean velocities requires a closed system of equations for the temporal evolution of these mean stresses.

In a recent paper [19], we identified the signature of the magnetorotational instability in the mean Maxwell and Reynolds stresses due to correlated fluctuations of the form $\delta \boldsymbol{v} = [\delta v_r(t,z), \delta v_\phi(t,z), 0]$ and $\delta \boldsymbol{B} = [\delta B_r(t,z), \delta B_\phi(t,z), 0]$ in an incompressible, cylindrical, differentially rotating flow, threaded by a mean vertical magnetic field, $\bar{B}_z$. We demonstrated that a number of properties of the mean stresses during the initial phase of exponential growth of the MRI are approximately preserved in the saturated state reached in local three-dimensional numerical simulations. As a first step in our calculation, we aim to derive dynamical equations for the mean stresses that describe this result.

In order to work with dimensionless variables we consider the characteristic time and length scales set by $1/\Omega_0$ and $\bar{v}_{\rm Az}/\Omega_0$. Here, Ω_0 and $\bar{v}_{\rm Az} = \bar{B}_z/\sqrt{4\pi\rho_0}$ denote the local values of the angular frequency and the (vertical) Alfvén speed in the disk with local density ρ_0 at the fiducial radius r_0. The equations governing the local dynamics of the (dimensionless) MRI-driven fluctuations in Fourier space, are then given by [19]

$$\partial_t \delta\hat{v}_r = 2\delta\hat{v}_\phi + ik_n\delta\hat{b}_r, \tag{2}$$

$$\partial_t \delta\hat{v}_\phi = (q-2)\delta\hat{v}_r + ik_n\delta\hat{b}_\phi, \tag{3}$$

$$\partial_t \delta\hat{b}_r = ik_n\delta\hat{v}_r, \tag{4}$$

$$\partial_t \delta\hat{b}_\phi = -q\delta\hat{b}_r + ik_n\delta\hat{v}_\phi, \tag{5}$$

where $\delta\hat{v}_i$ and $\delta\hat{b}_i$ stand for the Fourier transforms of the dimensionless physical fluctuations $\delta v_i/\bar{v}_{\rm Az}$ and $\delta B_i/\bar{B}_z$. The wave number, k_n, denotes the mode with n nodes in the vertical direction and the parameter $q \equiv -d\ln\Omega/d\ln r|_{r_0}$ is a measure of the local shear.

Using the fact that the modes with vertical wave vectors dominate the fast growth driven by the MRI, we obtain a set of equations to describe the initial exponential growth of the Reynolds and Maxwell stresses. We start from the equations for the fluctuations (2)–(5) and use the fact that the mean value of the product of two functions f and g, with zero means, is given by [19]

$$\langle fg\rangle(t) \equiv 2\sum_{n=1}^{\infty}{\rm Re}[\hat{f}(k_n,t)\hat{g}^*(k_n,t)]. \tag{6}$$

By combining different moments of Eqs. (2)–(5) we obtain the dimensionless set

$$\partial_t\bar{R}_{rr} = 4\bar{R}_{r\phi} + 2\bar{W}_{r\phi}, \tag{7}$$

$$\partial_t\bar{R}_{r\phi} = (q-2)\bar{R}_{rr} + 2\bar{R}_{\phi\phi} - \bar{W}_{rr} + \bar{W}_{\phi\phi}, \tag{8}$$

$$\partial_t\bar{R}_{\phi\phi} = 2(q-2)\bar{R}_{r\phi} - 2\bar{W}_{\phi r}, \tag{9}$$

$$\partial_t\bar{M}_{rr} = -2\bar{W}_{r\phi}, \tag{10}$$

$$\partial_t\bar{M}_{r\phi} = -q\bar{M}_{rr} + \bar{W}_{rr} - \bar{W}_{\phi\phi}, \tag{11}$$

$$\partial_t\bar{M}_{\phi\phi} = -2q\bar{M}_{r\phi} + 2\bar{W}_{\phi r}, \tag{12}$$

where we have defined the tensor $\bar{W}_{ij}$ with components

$$\bar{W}_{rr} \equiv \langle\delta v_r\delta j_r\rangle = \langle\delta b_\phi\delta\omega_\phi\rangle, \tag{13}$$

$$\bar{W}_{r\phi} \equiv \langle\delta v_r\delta j_\phi\rangle = -\langle\delta b_r\delta\omega_\phi\rangle, \tag{14}$$

$$\bar{W}_{\phi r} \equiv \langle\delta v_\phi\delta j_r\rangle = -\langle\delta b_\phi\delta\omega_r\rangle, \tag{15}$$

$$\bar{W}_{\phi\phi} \equiv \langle\delta v_\phi\delta j_\phi\rangle = \langle\delta b_r\delta\omega_r\rangle. \tag{16}$$

Here, δj_i and $\delta\omega_i$, for $i = r, \phi$, stand for the components of the induced current $\delta\boldsymbol{j} = \boldsymbol{\nabla}\times\delta\boldsymbol{b}$ and vorticity $\delta\boldsymbol{\omega} = \boldsymbol{\nabla}\times\delta\boldsymbol{v}$ fluctuations. Note that the components of the tensor $\bar{W}_{ij}$ are defined in terms of the correlations between the velocity and current fields. However, for the case under consideration, these are identical to the corresponding correlations between the magnetic field and vorticity fluctuations [Eqs. (13)–(16)].

Equations (7)–(12) show that the MRI-driven growth of the Reynolds and Maxwell tensors can be described formally *only* via the correlations $\bar{W}_{ij}$ that connect the equations for their temporal evolution. Note that, in contrast to the correlations $\langle\delta v_i\delta\omega_k\rangle$ and $\langle\delta b_i\delta j_k\rangle$, that appear naturally in helical dynamo modeling and transform as tensor densities (i.e., as the product of a vector and an axial vector), the correlations $\langle\delta v_i\delta j_k\rangle$ and $\langle\delta b_i\delta\omega_k\rangle$ transform as tensors. Moreover, the tensor $\bar{W}_{ij}$ cannot be recast in terms of the cross helicity $\bar{H}_{ij} \equiv \langle\delta v_i\delta b_j\rangle$ because, for the unstable MRI modes, the ratio $\bar{H}_{ij}/\bar{W}_{ij}$ approaches zero at late times. Neither can $\bar{W}_{ij}$ be expressed in terms of the turbulent electromotive force $\langle\delta\boldsymbol{v}\times\delta\boldsymbol{b}\rangle$, as the latter vanishes under our set of assumptions, implying that no mean magnetic field is generated.

In order to describe the MRI-driven exponential growth of the stresses in Eqs. (7)–(12) we need to write an additional set of dynamical equations for the evolution of the tensor $\bar{W}_{ij}$. Using appropriate combinations of different moments of Eqs. (2)–(5), we obtain

$$\partial_t\bar{W}_{rr} = q\bar{W}_{r\phi} + 2\bar{W}_{\phi r} + (\bar{k}^R_{r\phi})^2\bar{R}_{r\phi} - (\bar{k}^M_{r\phi})^2\bar{M}_{r\phi}, \tag{17}$$

$$\partial_t\bar{W}_{r\phi} = 2\bar{W}_{\phi\phi} - (\bar{k}^R_{rr})^2\bar{R}_{rr} + (\bar{k}^M_{rr})^2\bar{M}_{rr}, \tag{18}$$

$$\partial_t\bar{W}_{\phi r} = (q-2)\bar{W}_{rr} + q\bar{W}_{\phi\phi} + (\bar{k}^R_{\phi\phi})^2\bar{R}_{\phi\phi} - (\bar{k}^M_{\phi\phi})^2\bar{M}_{\phi\phi}, \tag{19}$$

$$\partial_t\bar{W}_{\phi\phi} = (q-2)\bar{W}_{r\phi} - (\bar{k}^R_{r\phi})^2\bar{R}_{r\phi} + (\bar{k}^M_{r\phi})^2\bar{M}_{r\phi}, \tag{20}$$

where we have defined the set of mean wave numbers

$$(\bar{k}^R_{ij})^2 \equiv \frac{\sum_{n=1}^{\infty}k_n^2{\rm Re}[\delta\hat{v}_i\delta\hat{v}_j{}^*]}{\sum_{n=1}^{\infty}{\rm Re}[\delta\hat{v}_i\delta\hat{v}_j{}^*]},$$

$$(\bar{k}^M_{ij})^2 \equiv \frac{\sum_{n=1}^{\infty}k_n^2{\rm Re}[\delta\hat{b}_i\delta\hat{b}_j{}^*]}{\sum_{n=1}^{\infty}{\rm Re}[\delta\hat{b}_i\delta\hat{b}_j{}^*]}.$$

The system of Eqs. (7)–(12) and (17)–(20) describes the temporal evolution of the stresses during the exponential growth of the MRI in a way that is formally correct, with no approximations. Motivated by the similarity between the ratios of the stresses during the exponential growth of the MRI and during the saturated turbulent state [19], we propose to use the right-hand sides of Eqs. (7)–(12) and (17)–(20) as a local model for the source of turbulence in MRI-driven magnetohydrodynamic flows. Of course, in the turbulent regime, the various average wave numbers $\bar{k}_{ij}^{R,M}$ will depend on the spectrum of velocity and magnetic field fluctuations. As the lowest order model for these wave numbers we choose

$$(\bar{k}_{ij}^{R,M})^2 = \zeta^2 k_{\max}^2 = \zeta^2\left(q - \frac{q^2}{4}\right) \quad \text{for} \quad i, j = r, \phi, \quad (21)$$

where ζ is a parameter of order unity and $k_{\max}$ corresponds to the wave number at which the growth rate of the *fluctuations* reaches its maximum value, $\gamma_{\max} \equiv q/2$.

By construction, the set of Eqs. (7)–(12) and (17)–(20) leads to the expected exponential growth of the mean stresses driven by modes with wave vectors perpendicular to the disk midplane. These modes, however, are known to be subject to parasitic instabilities, which transfer energy to modes in the perpendicular (k_r, k_ϕ) directions [20]. The initial fast growth experienced by the stresses will eventually be slowed down by the combined effects of nonlinear couplings between modes and of dissipation at the smallest scales of interest.

The terms accounting for these interactions would appear in the equations for the stresses as triple correlations between components of the velocity and magnetic fields; i.e., they would be of the form $\langle \delta v_i \delta v_j \delta b_k \rangle$ and $\langle \delta v_i \delta b_j \delta b_k \rangle$. These types of nonlinear terms have also been considered by Ogilvie [18], who proposed scalings of the form

$$\langle \delta v_i \delta v_j \delta b_k \rangle \sim \langle \delta v_i \delta v_j \rangle \langle \delta b_k \delta b_k \rangle^{1/2} \sim \bar{M}_{kk}^{1/2} \bar{R}_{ij}. \quad (22)$$

In the absence of a detailed model for these correlations and motivated again by the similarity of the stress properties during the exponential growth of the MRI and the saturated turbulent state [19], we introduce a phenomenological description of the nonlinear effects on the evolution of the various stresses. In particular, denoting by $\bar{X}_{ij}$ the ij component of any one of the three tensors, i.e., $\bar{R}_{ij}$, $\bar{M}_{ij}$, or $\bar{W}_{ij}$, we add to the equation for the temporal evolution of that component the sink term

$$\left.\frac{\partial \bar{X}_{ij}}{\partial t}\right|_{\text{sink}} \equiv -\sqrt{\frac{\bar{M}}{\bar{M}_0}} \bar{X}_{ij}. \quad (23)$$

Here, $\bar{M}/2 = (\bar{M}_{rr} + \bar{M}_{\phi\phi})/2$ is the mean magnetic energy density in the fluctuations and $\bar{M}_0$ is a parameter.

Adding the sink terms to the system of Eqs. (7)–(12) and (17)–(20) leads to a saturation of the stresses after a few characteristic time scales, *preserving* the ratio of the various stresses to the value determined by the exponential growth due to the MRI and characterized by the parameter ζ. These nonlinear terms dictate only the saturated level of the mean magnetic energy density according to $\lim_{t\to\infty} \bar{M} = \Gamma^2 \bar{M}_0$, where Γ is the growth rate for the *stresses*, which, in the case of a Keplerian disk, with $q = 3/2$, is given by $\Gamma^2 = 2[\sqrt{1 + 15\zeta^2} - (1 + 15\zeta^2/8)]$.

We infer the dependence of the energy density scale $\bar{M}_0/2$ on the four characteristic scales in the problem Ω_0, H (the vertical length of the box), ρ_0, and $\bar{v}_{Az}$ using dimensional analysis. We obtain $\bar{M}_0/2 \propto \rho_0 H^\delta \Omega_0^\delta \bar{v}_{Az}^{2-\delta}$, which leads, with the natural choice $\delta = 1$, to $\bar{M}_0/2 \equiv \xi \rho_0 H \Omega_0 \bar{v}_{Az}$, where we show the dimensional quantities explicitly and introduce the parameter ξ.

Our expression for $\bar{M}_0$ describes the same scaling between the magnetic energy density during saturation and the various parameters characterizing the disk found in a series of shearing box simulations threaded by a finite vertical magnetic field and with a Keplerian shearing profile [4]. By performing a numerical study of the late-time solutions of the proposed model, we found a unique set of values (ζ, ξ) such that its asymptotic limit describes the correlations found in these numerical simulations.

Figure 1 shows the correlations between the $r\phi$ components of the Maxwell and Reynolds stresses and the mean magnetic energy density found during the saturated state in numerical simulations [4]. In our model, this ratio depends only on the parameter ζ and the shear q (held fixed at $q = 3/2$ in the simulations). It is evident that, in the numerical simulations, the ratios of the stresses to the magnetic energy density are also practically independent of any of the initial parameters in the problem that determine the magnetic energy density during saturation (i.e., the x axis in the plot). This is indeed why we required our model of saturation to preserve the stress ratios that are determined

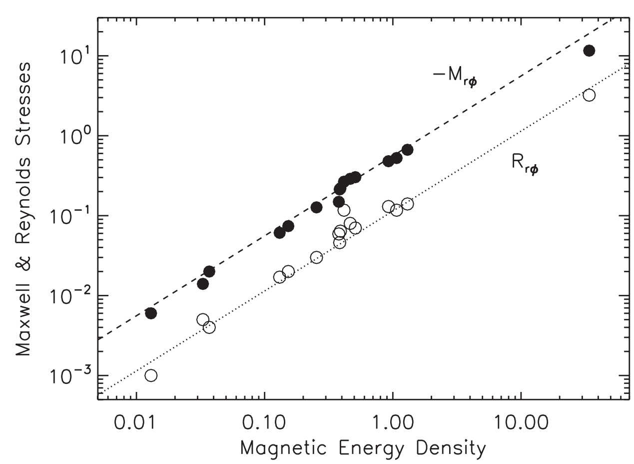

FIG. 1. Correlations between the Maxwell and Reynolds stresses and mean magnetic energy density at saturation in MRI-driven turbulent shearing boxes [4]. The lines show the result obtained with our model in the asymptotic limit for $\zeta = 0.3$.

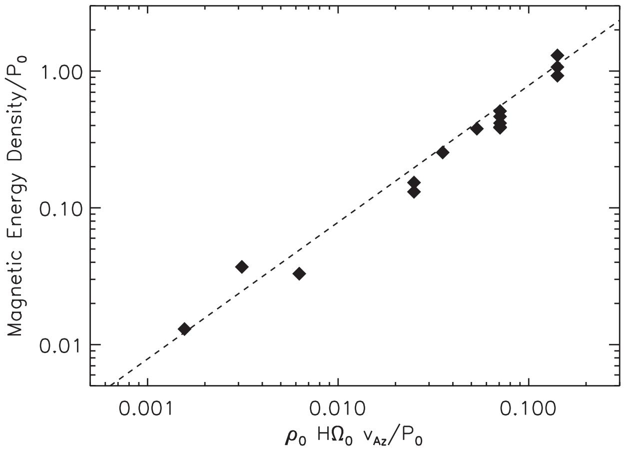

FIG. 2. The mean magnetic energy density in terms of a saturation predictor found at late times in numerical simulations of MRI-driven turbulence [4]. The line shows the result obtained with our model in the asymptotic limit for $\zeta = 0.3$ and $\xi = 11.3$.

during the exponential phase of the MRI. Assigning the same fractional uncertainty to all the numerical values for the stresses, our model describes both correlations simultaneously for $\zeta = 0.3$.

Figure 2 shows the mean magnetic energy density in terms of a saturation predictor found in numerical simulations [4]. For $\zeta = 0.3$ and assigning the same fractional uncertainty to all the numerical values for the stresses, we obtain the best fit for $\xi = 11.3$. These values for the parameters complete the description of our model.

In summary, in this Letter we developed a local model for the evolution and saturation of the Reynolds and Maxwell stresses in MRI-driven turbulent flows. The model is formally complete when describing the initial exponential growth and pumping of the MRI-driven stresses, and thus satisfies, by construction, all the mathematical requirements described by Ogilvie [18]. Although it is based on the absolute minimum physics (shear, uniform $\bar{B}_z$, 2D fluctuations) for the MRI to be at work, the model is able, in its asymptotic limit, to recover successfully the correlations found in three-dimensional local numerical simulations [4].

Finally, the local model described in this Letter contains an unexpected feature. The mean magnetic field in the vertical direction couples the Reynolds and Maxwell stresses via the correlations between the fluctuations in the velocity field and the fluctuating currents generated by the perturbations in the magnetic field. These second order correlations, which we denoted by the tensor $\bar{W}_{ij}$, play a crucial role in driving the exponential growth of the mean stresses and energy densities observed in numerical simulations. To our knowledge this is the first time that their relevance has been pointed out in either the context of dynamo theory or MRI-driven turbulence.

We thank Eric Blackman, Gordon Ogilvie, and Jim Stone for useful discussions. M. E. P. acknowledges the hospitality of the Institute for Advanced Study during part of this work. This work was partially supported by NASA Grant No. NAG-513374. M. E. P. is supported through a Jamieson Grant from the Astronomy Department at the University of Arizona.

*Electronic address: mpessah@as.arizona.edu

[1] N. I. Shakura and R. A. Sunyaev, Astron. Astrophys. **24**, 337 (1973).
[2] S. A. Balbus and J. C. B. Papaloizou, Astrophys. J. **521**, 650 (1999).
[3] S. A. Balbus and J. F. Hawley, Astrophys. J. **376**, 214 (1991); Rev. Mod. Phys. **70**, 1 (1998).
[4] J. F. Hawley, C. F. Gammie, and S. A. Balbus, Astrophys. J. **440**, 742 (1995); **464**, 690 (1996).
[5] T. Sano, S. Inutsuka, N. J. Turner, and J. M. Stone, Astrophys. J. **605**, 321 (2004).
[6] J. F. Hawley, Astrophys. J. **528**, 462 (2000); **554**, 534 (2001).
[7] J. M. Stone and J. E. Pringle, Mon. Not. R. Astron. Soc. **322**, 461 (2001).
[8] J. F. Hawley and J. H. Krolik, Astrophys. J. **548**, 348 (2001).
[9] M. Abramowicz, A. Brandenburg, and J. P. Lasota, Mon. Not. R. Astron. Soc. **281**, L21 (1996).
[10] P. J. Armitage, Astrophys. J. Lett. **501**, L189 (1998).
[11] F. Krause and K. H. Rädler, *Mean-Field Magnetohydrodynamics and Dynamo Theory* (Pergamon, New York, 1980).
[12] A. Brandenburg and K. Subramanian, Phys. Rep. **417**, 1 (2005).
[13] E. G. Blackman, Mon. Not. R. Astron. Soc. **323**, 497 (2001).
[14] A. Brandenburg, Astron. Nachr. **326**, 787 (2005).
[15] E. G. Blackman and G. B. Field, Phys. Rev. Lett. **89**, 265007 (2002).
[16] S. Kato and A. Yoshizawa, Publ. Astron. Soc. Jpn. **45**, 103 (1993); **47**, 629 (1995).
[17] S. Kato, J. Fukue, and S. Mineshige, *Black Hole Accretion Disks* (Kioto University Press, Kyoto, Japan, 1998).
[18] G. I. Ogilvie, Mon. Not. R. Astron. Soc. **340**, 969 (2003).
[19] M. E. Pessah, C. K. Chan, and D. Psaltis, Mon. Not. R. Astron. Soc. **372**, 183 (2006).
[20] J. Goodman and G. Xu, Astrophys. J. **432**, 213 (1994).