Artificial Electric Field in Fermi Liquids

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Based on the Keldysh formalism, we derive an effective Boltzmann equation for a quasiparticle constrained within a particular Fermi surface in an interacting Fermi liquid. This provides a many-body derivation of Berry curvatures in electron dynamics with spin-orbit coupling, which has received much attention in recent years in noninteracting models. As is well known, the Berry curvature in momentum space modifies naïve band dynamics via an ''artificial magnetic field'' in momentum space. Our Fermi liquid formulation completes the reinvention of modified band dynamics by introducing in addition an *artificial electric field*, related to Berry curvature in frequency and momentum space. We show explicitly how the artificial electric field affects the renormalization factor and transverse conductivity of interacting $U(1)$ Fermi liquids with nondegenerate bands.

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Recent experimental developments in spintronics, notably several measurements of the evidently *intrinsic* anomalous Hall effect in ferromagnetic metals [1] and observations of the spin Hall effect in semiconductors [2,3], have highlighted the importance of intrinsic spin-orbit effects in conductors. Theoretically, there has been a renewal of interest in the fundamental dynamics of Bloch electrons in periodic solids taking into account spin-orbit interactions, following ideas pioneered by Karplus and Luttinger [4] and elegantly formulated by Sundaram and Niu [5]. In particular, in a noninteracting band picture, the low-energy Hilbert space is restricted to states in the conduction band containing the Fermi surface at each quasimomentum k along this surface. This local constraint, like other local constraints in physics [6], generates a gauge symmetry in momentum space, corresponding to the freedom to redefine Bloch functions by a momentum-dependent phase factor [or $SU(2)$ matrix in the case of twofold degenerate bands]. Consequently, a wave packet analysis of noninteracting Bloch states shows that this structure induces an artificial magnetic field (AMF), which contributes a Lorentz force in k space, mathematically analogous to the action of the real magnetic field $\mathbf{b}$ in real R space [5]. The AMF $\boldsymbol{\mathcal{B}}_\alpha$ and its associated magnetic gauge field $\mathcal{A}_\alpha$ are defined from the periodic part of the Bloch wave function of conduction bands $|u_\alpha\rangle$: $\mathcal{B}_{\alpha,i} \equiv i\epsilon_{ijm}\partial_{k_j}\mathcal{A}_{\alpha,m}$ and $\mathcal{A}_{\alpha,i} \equiv \langle u_\alpha|\partial_{k_i}u_\alpha\rangle$. The fundamental framework of Fermi liquid theory suggests that these conclusions persist in realistic metals, in which the Coulomb interactions between electrons are not weak. Haldane has argued that this is indeed the case for quasiparticles at the Fermi surface, with $|u_\alpha\rangle$ replaced by the eigenvector of the inverse of the Hermitian part of the retarded (advanced) Green function [7]. In this Letter, we identify an additional manifestation of this gauge structure unique to interacting Fermi liquids. In particular, because the renormalized eigenstates thus introduced are, in general, dependent on frequency ω, it is natural to introduce an artificial electric field (AEF) and electrostatic field as $\mathcal{E}_{\alpha,i} \equiv i(\partial_\omega \mathcal{A}_{\alpha,i} - \partial_{k_i}\mathcal{A}_{\alpha,0})$ and $\mathcal{A}_{\alpha,0} \equiv \langle u_\alpha|\partial_\omega u_\alpha\rangle$, respectively. These quantities vanish in the absence of interactions, since the ω dependence of the quasiparticle states arises from that of the (collisional) self-energy.

In the following, we present explicit calculations that demonstrate that this AEF indeed has physical effects. In addition, our Keldysh formulation provides a many-body derivation of an effective kinetic equation embodying the entire emergent gauge structure and a systematic procedure for actually carrying out the projection into the low-energy band. We concentrate on metallic materials lacking either inversion or time-reversal symmetry, in which case the bands are nondegenerate and the gauge structure is $U(1)$. For such a $U(1)$ Fermi liquid (FL), we demonstrate that both AMF and AEF, which are estimated ''on-shell'' [see Eq. (12) in combination with Eqs. (7) and (9) for their more precise definitions], in fact enter into the effective equation of motion (EOM) for a quasiparticle:

$$\begin{aligned} \frac{dR}{dT} &= \mathbf{v} + (\boldsymbol{\mathcal{B}}_\alpha - \boldsymbol{\mathcal{E}}_\alpha \times \mathbf{v}) \times \frac{dk}{dT}, \\ \frac{dk}{dT} &= -\mathbf{e} + \mathbf{b} \times \frac{dR}{dT}. \end{aligned} \tag{1}$$

$\boldsymbol{\mathcal{E}}_\alpha$ thus provides another source of the Lorentz force in k space, in addition to the contribution of the $\boldsymbol{\mathcal{B}}_\alpha$ field identified in Ref. [5]. It therefore modifies the transverse current carried by a quasiparticle and, hence, the intrinsic anomalous Hall conductivity of a conducting ferromagnet to

$$\sigma_{jk} = \epsilon_{jkl}\frac{e^2}{\hbar}\sum_\alpha \int \frac{dk}{(2\pi)^d}(\boldsymbol{\mathcal{B}}_\alpha - \boldsymbol{\mathcal{E}}_\alpha \times \mathbf{v}_\alpha)_l n_\alpha, \tag{2}$$

where the sum α is over bands, and n_α is the distribution function. In particular, Eq. (2) can also be proven to be consistent with the many-body formulation of the trans-

verse conductivity [7,8], more directly by using Ward identities which relate the current vertex correction to single quasiparticle Green functions [9]. As another physical consequence of these artificial electromagnetic fields (AEF or AMF), we will also show that the linear response of the momentum-resolved single-particle density of states to applied electromagnetic fields is characterized by these artificial fields. Specifically, in the presence of $\mathbf{e}$ and $\mathbf{b}$, the quasiparticle renormalization factor of a $U(1)$ FL acquires the following topological corrections:

$$Z' = 1 - \tfrac{1}{2}(\boldsymbol{\mathcal{B}}_\alpha \cdot \mathbf{b} + \boldsymbol{\mathcal{E}}_\alpha \cdot \mathbf{e}). \tag{3}$$

This, in principle, makes it possible to detect the distribution of these ''artificial'' fields in the momentum space, using angle resolved photoemission spectroscopy (ARPES) experiments.

We now turn to the derivation of the above results and the gauge-invariant effective kinetic equation. We begin with a quite general semimicroscopic model in which the electronic spectrum is described by a $k \cdot p$-type expansion about some (arbitrary) point in the Brillouin zone. One may keep as many bands as are deemed close enough in energy to be relevant to the physics, and our arguments do not depend upon the order in the expansion in k. Familiar examples would be the multiband Luttinger models for commonly studied semiconductors, in which the natural expansion is about the Γ point. In principle, one should choose the expansion point so that there are no low-energy band crossings intervening between it and the Fermi surface, and there could be difficulties in treating open Fermi surfaces with the periodic structure of the quasimomentum. However, while a fully microscopic treatment would be desirable, our approach is nevertheless quite general, and, moreover, we speculate that our final results naturally extend and apply to arbitrary $U(1)$ FLs.

The advantage of this formulation is that we can Fourier transform in the usual way to a continuous real space coordinate r, i.e., $k \to -i\nabla_r$. The noninteracting Hamiltonian is thereby expressed in terms of the fermionic ''envelope fields'' $\psi_\alpha(r)$: $\mathcal{H}_0 = \sum_{\alpha,\alpha'} \int dr \psi^\dagger_\alpha(r) \times [\hat{H}_0(-i\nabla_r, r)]_{\alpha\alpha'} \psi_{\alpha'}(r)$, where the band index $\alpha^{(\prime)}$ runs from 1 to N_b. Specific information about lattice, orbital, and spin-orbit couplings is encoded into the matrix structure of $[\hat{H}_0]$. We will employ a natural form of the interaction, $\mathcal{H}_1 = \sum \iint \psi^\dagger_{\alpha_1}(r_1)\psi^\dagger_{\alpha_2}(r_2) V_{\alpha_1\alpha_2\alpha'_2\alpha'_1}(r_1, r_2) \times \psi_{\alpha'_2}(r_2)\psi_{\alpha'_1}(r_1)$, though the form of our results does not depend in detail upon this.

Bearing in mind this Hamiltonian $\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1$, we can proceed with the Keldysh formalism, beginning with an analysis of the spectral function $[\hat{\mathsf{A}}(r, t|r', t')]_{(\alpha|\alpha')} = \langle\{\psi_\alpha(r, t), \psi^\dagger_{\alpha'}(r', t')\}_+\rangle$. It obeys [10,11]

$$[\hat{G}_0^{-1} - \hat{\Sigma}^{\mathrm{HF}} - \hat{\sigma}, \hat{\mathsf{A}}]_{\otimes,-} \equiv [\hat{L}, \hat{\mathsf{A}}]_{\otimes,-} = [\hat{\Gamma}, \mathrm{Re}\hat{G}]_{\otimes,-} \to \hat{0}. \tag{4}$$

Here the product $\otimes$ denotes the convolution with respect to time t, space r, and band index α, $(\hat{B} \otimes \hat{C})(1|1') \equiv \int d\bar{1}\, \hat{B}(1|\bar{1}) \cdot \hat{C}(\bar{1}|1')$. The commutator here is defined by using this convolution: $[\hat{B}, \hat{C}]_{\otimes,-} \equiv \hat{B} \otimes \hat{C} - \hat{C} \otimes \hat{B}$. The bare Green function is as usual defined by $\hat{G}_0^{-1}(1|1') \equiv [i\partial_{t_1}\hat{1} - \hat{H}_0(-i\nabla_{r_1}, r_1)]_{\alpha_1\alpha'_1}\delta(t_1 - t'_1)\delta(r_1 - r'_1)$. $\hat{\Sigma}^{\mathrm{HF}}(1|1')$ and $\hat{\sigma}(1|1')$ are the Hartree-Fock part (temporally instantaneous) and the Hermitian part, respectively, of the collisional self-energy (noninstantaneous), while $\hat{\Gamma}(1|1')$ is the anti-Hermitian part of the collisional self-energy. In a FL, the decay rate represented by the $[\hat{\Gamma}, \mathrm{Re}\hat{G}]_{\otimes,-}$ in Eq. (4) is expected to be $O((\omega - \mu)^2, T^2)$, due to the phase-space constraints inherent to FL theory. Because we focus on low temperature physics [and indeed require the self-energy only to $O(\omega - \mu)$, the order that determines the nonvanishing electric field at the Fermi surface [12]], we henceforth ignore this lifetime effect, replacing the right-hand side in Eq. (4) as indicated.

Even this dissipationless Keldysh equation is still nontrivial and involved, due to the presence of the band index, whose effect is the central issue of this work. In the following, we will argue a general method of projecting out the irrelevant band indices and how to obtain the reduced Keldysh (kinetic) equation only for the relevant bands.

The first step is to apply the Wigner transformation, so that frequency and momentum (ω and q) are introduced by the Fourier transformation for the relative coordinate in space and time $(t, r) = (t_1 - t_{1'}, r_1 - r_{1'})$. Namely, $\hat{B}(\omega, q; T, R) = \int dr dt \hat{B}(r_1, t_1|r_{1'}, t_{1'}) e^{-iqr + i\omega t}$, where the center of mass coordinate in space and time $X = (T, R) \equiv [(t_1 + t_{1'})/2, (r_1 + r_{1'})/2]$ parametrize phase space in combination with $Q = (\omega, q)$. Then, following Ref. [10], the Keldysh equation in terms of Q and X becomes

$$-[\hat{L}, \hat{\mathsf{A}}]_- = \frac{i}{2}[\partial_{X_j}\hat{L}, \partial_{Q_j}\hat{\mathsf{A}}]_+ - \frac{1}{8}([\partial_{X_j}\partial_{X_k}\hat{L}, \partial_{Q_j}\partial_{Q_k}\hat{\mathsf{A}}]_- - \{X_k \leftrightarrow Q_k\}) - \{X_j \leftrightarrow Q_j\}. \tag{5}$$

The j and k summation over 0 to d are implicit from now on (d is the spatial dimension of our system). The (anti)-commutator here is defined in the sense of the product only with respect to band indices. The nonlocal correlation effect which was previously encoded in the space-time convolution is now partially taken into account via the so-called gradient expansion in $\partial_Q\partial_X \equiv -\partial_\omega\partial_T + \partial_{q_j}\partial_{R_j}$. The expansion is justified close to equilibrium, since all quantities are independent of X in equilibrium. We carry out the gradient expansion to 2nd order, at which $\boldsymbol{\mathcal{E}}$ and $\boldsymbol{\mathcal{B}}$ appear in the kinetic equation [see, for example, Eq. (14)].

The Wigner-transformed Lagrangian $\hat{L}$, when diagonalized, $\hat{L}_d = \hat{U}^\dagger \cdot \hat{L} \cdot \hat{U}$, defines a renormalized energy dispersion by its eigenvalue and renormalized Bloch functions by its eigenbasis. Namely, in this new basis, Eq. (5) at equilibrium can be satisfied by those spectral functions hav-

ing no off-diagonal components. The diagonal elements of $\hat{A} \equiv \hat{U}^\dagger \cdot \hat{\mathsf{A}} \cdot \hat{U}$ are verified *a posteriori* to become a delta function whose argument is the eigenvalue of $\hat{L}$

$$A_\alpha \equiv A_{\alpha\alpha} = \delta(L_{d,\alpha}) = \delta(\omega - E_\alpha(\omega, q)), \tag{6}$$

where $E_\alpha(\omega, q)$ is the αth eigenenergy of $\hat{H}_0 + \hat{\Sigma}^{\mathrm{HF}} + \hat{\sigma}$.

Out of equilibrium, the off-diagonal element also acquires small but finite weight, which we will take into account in a systematic way below. In this new basis, the simple derivatives in Eq. (5) are replaced by covariant ones:

$$\begin{aligned}
&\hat{U}^\dagger \cdot (\partial_{Q_j} \hat{L}) \cdot \hat{U} = \partial_{Q_j} \hat{L}_d + \hat{\mathcal{A}}_{Q_j} \cdot \hat{L}_d - \hat{L}_d \cdot \hat{\mathcal{A}}_{Q_j}, \\
&\hat{U}^\dagger \cdot (\partial_{X_j} \hat{A}) \cdot \hat{U} = \partial_{X_j} \hat{A} + \hat{\mathcal{A}}_{X_j} \cdot \hat{A} - \hat{A} \cdot \hat{\mathcal{A}}_{X_j}, \\
&\hat{U}^\dagger \cdot [\hat{L}, \hat{\mathsf{A}}]_- \cdot \hat{U} = [\hat{L}_d, \hat{A}]_-, \qquad \hat{\mathcal{A}}_{Q_j} \equiv \hat{U}^\dagger \partial_{Q_j} \hat{U},
\end{aligned} \tag{7}$$

N_b times N_b matrices $\hat{\mathcal{A}}_{Q_j}$ and $\hat{\mathcal{A}}_{X_j}$ are the renormalized gauge field, through which we will define AEF or AMF later [see Eqs. (9) and (12)]. As the off-diagonal components of the spectral function are expected to be small in this new basis, they may be eliminated in favor of the diagonal components. Specifically, this occurs because the Keldysh equation for the $A_{\alpha\beta}$ ($\alpha \neq \beta$) contains the term $\Delta_{\alpha\beta} A_{\alpha\beta}$, with the direct band gap $\Delta_{\alpha\beta} = E_\alpha - E_\beta$ as the sole nonvanishing term at 0th order in the gradient expansion—forcing $A_{\alpha\beta}$ to vanish in equilibrium. Accordingly, $A_{\alpha\beta}$ is of at most 1st order in the gradient expansion, and it can be solved for in terms of the diagonal elements A_α by iteration. By contrast, the diagonal components of the Keldysh equation appear all at first and higher order in the gradient expansion. Substituting the iterative solution for $A_{\alpha\beta}$ into these diagonal equations, we can obtain a set of N_b equations for the diagonal parts $A_\alpha = A_{\alpha\alpha}$ alone, to the desired (2nd) order in gradients.

Finally, we focus on a specific band "α" which contains a Fermi surface. In the low frequency region ($|\omega - E_\alpha| \simeq |\omega - \mu| \ll \min_\beta |\Delta_{\alpha\beta}|$), the diagonal elements of the spectral functions for the other bands are expected to have negligible weight. This is because in equilibrium, in the noninteracting limit, $A_{k\neq i}$ is sharply peaked at an energy separated from the chemical potential by the direct band gap. With interactions, or out of equilibrium, there may be some incoherent weight around $\omega \simeq \mu$, but it is expected to be extremely small. Thus, the other bands can also be eliminated, so that we obtain independent Keldysh equations for the diagonal spectral functions.

Following this prescription, we obtain the differential equation $\mathcal{L}^{(1)}(A_\alpha) + \mathcal{L}^{(2)}(A_\alpha) = 0$, with [9]

$$\begin{aligned}
\mathcal{L}^{(1)}(A_\alpha) &= [\partial_{X_j}(L_{d,\alpha} - \mathcal{M}_\alpha)](\partial_{Q_j} A_\alpha) - \{X_j \leftrightarrow Q_j\}, \\
\mathcal{L}^{(2)}(A_\alpha) &= \tfrac{1}{4}(\partial_{X_j} L_{d,\alpha})(\partial_{Q_j} \Omega^\alpha_{X_k Q_k}) A_\alpha \\
&\quad - (\partial_{X_k} L_{d,\alpha}) \Omega^\alpha_{Q_k X_j} (\partial_{Q_j} A_\alpha) - \{X_j \leftrightarrow Q_j\} \\
&\quad - \{X_k \leftrightarrow Q_k\} + \{X_k, X_j \leftrightarrow Q_k, Q_j\}.
\end{aligned} \tag{8}$$

Here the Berry curvatures associated with the αth band, i.e., $\Omega^\alpha_{Q_j X_k}$ and its 3 counterparts, are defined in terms of the renormalized gauge fields introduced in Eq. (7), e.g.,

$$\Omega^\alpha_{Q_j X_k} \equiv -i \sum_{\beta \neq \alpha} [\hat{\mathcal{A}}_{Q_j}]_{\alpha\beta} [\hat{\mathcal{A}}_{X_k}]_{\beta\alpha} + \text{c.c.} \tag{9}$$

$\mathcal{M}_\alpha$ in Eq. (8) denotes an energy correction of the 1st order gradient expansion: $\mathcal{M}_\alpha \equiv \frac{i}{2} \sum_{j=0}^{d} \sum_{\beta\neq\alpha} [\hat{\mathcal{A}}_{X_j}]_{\alpha\beta} \cdot (L_{d,\alpha} - L_{d,\beta}) \cdot [\hat{\mathcal{A}}_{Q_j}]_{\beta\alpha} + \text{c.c.}$ The Zeeman coupling energy between orbital momentum and external magnetic field, which was previously found in the noninteracting case [5], is encoded into the $\sum_{j=1}^{d}$ terms in this energy correction. On the other hand, the temporal term, i.e., $j = 0$, describes a coupling of the electric dipole moment of quasiparticles to the external electric field, which is unique to an interacting FL [9].

We can now extract the low-energy behavior of the spectral function. The renormalized energy dispersion of the αth band $\epsilon(q; X)$ is determined from

$$L_{d,\alpha} - \mathcal{M}_\alpha \equiv \epsilon - E_\alpha(\epsilon, q; X) - \mathcal{M}_\alpha(\epsilon, q; X) = 0. \tag{10}$$

Indeed, neglecting $\mathcal{L}^{(2)}$, the form $A_\alpha = \delta(L_{d,\alpha} - \mathcal{M}_\alpha)$ satisfies the Keldysh equation to the 2nd order in the gradient expansion. Therefore, without $\mathcal{L}^{(2)}$, the spectral function is sharply peaked at $\omega = \epsilon$, with a weight given by the conventional "renormalization factor" $Z = [\partial_\omega (L_{d,\alpha} - \mathcal{M}_\alpha)]^{-1}_{|\omega=\epsilon}$. However, due to the presence of $\mathcal{L}^{(2)}$, which contains terms proportional to A_α rather than its derivative, the solution up to the 2nd order of the gradient expansion acquires an additional renormalization factor $Z' = 1 - \frac{1}{2}(\Omega^\alpha_{X_k Q_k})_{|\omega=\epsilon}$, and

$$A_\alpha = Z' \delta(L_{d,\alpha} - \mathcal{M}_\alpha) = Z Z' \delta(\omega - \epsilon). \tag{11}$$

To unmask the meaning of this new term Z', let us specialize to the situation where the disequilibrium is generated by a physical (i.e., external) electromagnetic gauge field $(\mathrm{a}_0, \mathbf{a})$, with corresponding physical electromagnetic fields $\mathbf{b} = \boldsymbol{\nabla} \times \mathbf{a}$, $\mathbf{e} = \boldsymbol{\nabla}_R \mathrm{a}_0 - \partial_T \mathbf{a}$. In this case, energy $\epsilon[q, X] = \epsilon_0[q + \mathbf{a}(X)] + \mathrm{a}_0(X)$, where ϵ_0 is the (renormalized) band energy in equilibrium. Then, introducing canonical momentum and frequency $k = q + \mathbf{a}(T, R)$ and $\omega' \equiv \omega - \mathrm{a}_0(R)$ [13], Eq. (11) becomes $A_\alpha = Z Z' \delta(\omega' - \epsilon_0(k))$. Correspondingly, the Berry curvature appearing in the additional renormalization factor Z' is expressed in terms of the following artificial electromagnetic fields (AEF or AMF) estimated at $\omega = \epsilon$:

$$(\Omega^\alpha_{k_m k_n})_{|\omega=\epsilon} = \epsilon_{mnj} \mathcal{B}_{\alpha,j}, \qquad (\Omega^\alpha_{\omega k_j})_{|\omega=\epsilon} = \mathcal{E}_{\alpha,j}. \tag{12}$$

Namely, Z' simply reduces to an inner product between real electromagnetic fields and these AEF or AMF, as in Eq. (3). This result indicates a renormalization of the "angle resolved density of states" when both physical and AEF or AMF are present. A similar modification (dependent upon $\mathcal{B}_\alpha \cdot \mathbf{b}$ only) was proposed in Ref. [14].

This theoretical observation leads us to propose the momentum-resolved detection of AEF or AMF by ARPES, by observing the modification of the spectral weight by small applied electromagnetic fields. This is no doubt quite difficult in practice but demonstrates that the systematic measurement of the artificial fields is possible in principle.

We now turn from the spectral function to the quasiparticle dynamics, by deriving the effective Boltzmann equation for quasiparticles in a $U(1)$ FL. We start with the observation that the dissipationless Keldysh equation in Eq. (5) holds also for the lesser Green's function $\mathsf{g}^<(1,1') \equiv i\langle\psi^\dagger_{\alpha'}(r',t')\psi_\alpha(r,t)\rangle$ under the same assumptions [11]. Applying the same unitary transformation as above, i.e., $\hat{g}^< \equiv \hat{U}^\dagger \cdot \hat{\mathsf{g}}^< \cdot \hat{U}$, one then obtains exactly the same reduced Keldysh equation for the (α,α) component of $\hat{g}^<$ as in Eq. (8). This lesser Green's function may be decomposed according to $g^<_\alpha(Q;X) = iA_\alpha(Q;X)f_\alpha(Q;X)$, where $f_\alpha(Q;X)$ is a generalized Fermi distribution [11]. Inserting this into the reduced Keldysh equation thus obtained, we can readily obtain the EOM for f_α:

$$0 = ZZ'\delta(\omega-\epsilon)\{\partial_{X_j}(L_{d,\alpha}-\mathcal{M}_\alpha)\partial_{Q_j}f_\alpha - \partial_{X_k}\Omega^\alpha_{Q_kX_j}\partial_{Q_j}f_\alpha + \partial_{Q_k}\Omega^\alpha_{X_kX_j}\partial_{Q_j}f_\alpha - (X_j \leftrightarrow Q_j)\}.$$

Because of the sharp peak in $A_\alpha \simeq \delta(\omega-\epsilon)$, when integrating this EOM with respect to ω, we get an effective Boltzmann equation for a quasiparticle occupation number in q-R space; $n_\alpha(q{:}R,T) \equiv f_\alpha(q,\epsilon{:}R,T)$,

$$[1-\Omega^\alpha_{T\epsilon}+(\partial_{R_j}\epsilon)\Omega^\alpha_{q_j\epsilon}-(\partial_{q_j}\epsilon)\Omega^\alpha_{R_j\epsilon}]\partial_T n_\alpha = [\partial_{R_j}\epsilon+(\partial_T\epsilon)\Omega^\alpha_{\epsilon R_j}-(\partial_{R_k}\epsilon)\Omega^\alpha_{q_kR_j}+\Omega^\alpha_{TR_j}+(\partial_{q_k}\epsilon)\Omega^\alpha_{R_kR_j}]\partial_{q_j}n_\alpha - \{q_j\leftrightarrow R_j\}. \tag{13}$$

Note that, although all of the curvature terms in the above are now "on-shell" quantities, the partial q, R, and T derivatives encoded there apply only on their explicit dependence and do *not* apply on their arguments of ϵ.

In the presence of external electromagnetic fields, this effective Boltzmann equation can be substantially simplified in terms of the canonical momentum introduced above:

$$0 = \partial_T n_\alpha + [-\mathbf{e} + \mathbf{b}\times\mathbf{v} + \mathbf{b}\times(\mathbf{e}\times\boldsymbol{\mathcal{B}}'_\alpha) - \mathbf{b}\times((\mathbf{b}\times\mathbf{v})\times\boldsymbol{\mathcal{B}}'_\alpha)]\cdot\partial_k n_\alpha + [\mathbf{v}+\mathbf{e}\times\boldsymbol{\mathcal{B}}'_\alpha - (\mathbf{b}\times\mathbf{v})\times\boldsymbol{\mathcal{B}}'_\alpha]\cdot\partial_R n_\alpha, \tag{14}$$

where we used abbreviated notations $\boldsymbol{\mathcal{B}}'_\alpha \equiv \boldsymbol{\mathcal{B}}_\alpha - \boldsymbol{\mathcal{E}}_\alpha\times\mathbf{v}$ and $\mathbf{v} \equiv \partial_k\epsilon$ [15]. The partial X derivative of n_α was taken with the canonical momentum k fixed. By equating this with the continuity equation in phase space, $0 = \partial_T n_\alpha + (\partial_T k)\cdot\partial_k n_\alpha + (\partial_T R)\cdot\partial_R n_\alpha$, we obtain the quasiparticle equation of motion [Eq. (1)] to the accuracy of the 2nd order gradient expansion. Furthermore, extracting the conserved current from the continuity equation in zero external field ($\mathbf{b}=0$), one finds $\mathbf{j}_\alpha = \int_k[\mathbf{v}+\mathbf{e}\times\boldsymbol{\mathcal{B}}'_\alpha]n_\alpha$. The usual solution of the Boltzmann equation in linear response therefore gives immediately the anomalous Hall conductivity in Eq. (2).

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[12] As a simple proof-of-principle example showing a finite AEF, we have considered a lattice-regularized Rashba model under applied magnetic field, in which electrons are coupled with an Einstein phonon with characteristic frequency ω_0. At low temperature, only the Hermitian self-energy is non-negligible at low frequency. One finds a nonzero $\partial\hat{\sigma}/\partial\omega_{|\omega=\mu}$, which yields a finite AEF that can be directly calculated. For reasonable parameters and not too weak electron-phonon coupling, it is significant and substantially modifies the anomalous Hall effect of the noninteracting system [9].
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[15] Using $\hat{U}' \equiv \hat{U}_{|\omega=\epsilon}$, one could also express $\boldsymbol{\mathcal{B}}'_\alpha$ into a "magnetic-field-like" form: $\mathcal{B}'_{\alpha,j} \equiv i\epsilon_{jlm}d_{k_l}\mathcal{A}'_{\alpha,m}$, where $\mathcal{A}'_{\alpha,i} \equiv [\hat{U}'^\dagger d_{k_i}\hat{U}']_{\alpha\alpha}$. As in Eq. (3), however, the renormalization factor also acquires the topological correction characterized by the inner product between $(\mathbf{e},\mathbf{b})$ and $(\boldsymbol{\mathcal{E}}_\alpha,\boldsymbol{\mathcal{B}}_\alpha)$. Thus, we believe that $\boldsymbol{\mathcal{E}}_\alpha$ and $\boldsymbol{\mathcal{B}}_\alpha$ are a more natural set of fundamental quantities in FL theory than $\boldsymbol{\mathcal{B}}'_\alpha$ is.