Extreme Events in Deterministic Dynamical Systems

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The principal signatures of a deterministic dynamics in the statistical properties of extreme events are identified. Explicit expressions are derived for generic classes of dynamical systems giving rise to quasiperiodic, strongly chaotic, and intermittent chaotic behaviors.

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A typical record generated by a natural, technological, or societal system consists of periods where a relevant variable undergoes small scale fluctuations around a well-defined level provided by the long term average of the values available, interrupted by abrupt excursions to values that differ significantly from this level. Such *extreme events* are of paramount importance in a variety of contexts since they can signal phenomena such as the breakdown of a mechanical structure, an earthquake, a severe thunderstorm, flooding, or a financial crisis [1]. Information on the probability of their occurrence and the capability to predict the time and place at which this occurrence may be expected is thus of great value in, among others, the construction industry or the assessment of risks. While the probability of such events decreases with their magnitude, the damage that they may bring increases rapidly with the magnitude as does the cost of protection against them. These opposing trends make the task of prediction extremely challenging.

There exists a powerful statistical theory of extremes [2]. In its classical version, it deals with independent identically distributed random variables, and, in the asymptotic limit of infinite time observational window, it can be formulated entirely in terms of three universal types of probability functions: the Gumbel, Fréchet, and Weibull distributions. This conclusion holds true for correlated sequences as well, as long as the time autocorrelation function falls sufficiently fast to zero [3]. On the other hand, the fundamental laws of nature are deterministic. It is by now well established that deterministic dynamics is at the basis of a wide variety of complex nonlinear phenomena encountered at different levels of observation, in the form of abrupt transitions, a multiplicity of states, or spatiotemporal chaos. The objective of this Letter is to outline a theory of extremes for deterministic systems and to identify its principal signatures with respect to the classical statistical approach. Our approach will build on the possibility to extract probabilistic properties from the deterministic dynamics without resorting to coarse-graining, provided that the dynamics is sufficiently complex to possess some minimal ergodic properties [4,5]. Accordingly, a continuous (fine-grained) description will be used throughout.

The basic question asked in a problem of extremes is, given a sequence $X_0, \ldots, X_{n-1}$ of successive values of an observable monitored at regularly spaced times 0, $\tau, \ldots, (n-1)\tau$, what is the (cumulative) probability distribution F_n of the largest value x found, $M_n = \max(X_0, \ldots, X_{n-1})$; see, e.g., Ref. [2]:

$$F_n(x) = \mathrm{Prob}(X_0 \leq x, \ldots, X_{n-1} \leq x), \qquad a \leq x \leq b, \quad (1)$$

with $F_n(a) = 0$, $F_n(b) = 1$ and the associate probability density $\rho_n(x)$ deduced by differentiation of $F_n(x)$. Now, by definition, the multivariate probability density to realize the sequence $X_0, \ldots, X_{n-1}$ [not to be confused with the aforementioned $\rho_n(x)$] is

$$\rho_n(X_0, \ldots, X_{n-1}) = (\text{Prob to be in } X_0 \text{ in the first place}) \times \prod_{k=1}^{n-1} (\text{Prob to be in } X_{k-1} \text{ given one was in } X_0\ k \text{ time units before}). \quad (2)$$

The first factor in Eq. (2) is given by the invariant probability density $\rho(X_0)$. This quantity is smooth as long as the underlying system, be it stochastic or deterministic, possesses sufficiently strong ergodic properties. In contrast, the two classes differ by the nature of the conditional probabilities inside the n-fold product. While in stochastic systems these quantities are typically smooth, in deterministic dynamical systems they are Dirac delta functions $\delta(X_{k\tau} - f^{k\tau}(X_0))$, where $f(X_0)$ stands for the formal solution of the evolution equations and the upper index denotes the order in which the corresponding operator needs to be iterated.

By definition, the cumulative probability distribution $F_n(x)$ in Eq. (1)—the relevant quantity in a theory of extremes—is the n-fold integral of Eq. (2) over $X_0, \ldots, X_{n-1}$ from a up to the level x of interest. This converts the delta functions into Heaviside theta functions, yielding

$$F_n(x) = \int_a^x dX_0 \rho(X_0) \times \theta(x - f^\tau(X_0)) \ldots \theta(x - f^{(n-1)\tau}(X_0)). \quad (3)$$

In other words, $F_n(x)$ is obtained by integrating $\rho(X_0)$ over those ranges of X_0 in which $x \geq \{f^\tau(X_0), \ldots, f^{(n-1)\tau}(X_0)\}$. As x is moved upwards, new integration ranges will thus be added, since the slopes of the successive iterates $f^{k\tau}$ with respect to X_0 are, typically, both different from each other and X_0 dependent. Each of these ranges will open up past a threshold value where either the values of two different iterates will cross or an iterate will cross the manifold $x = X_0$. This latter type of crossing will occur at x values belonging to the set of periodic orbits of all periods up to $n-1$ of the dynamical system.

These observations entail that in a deterministic system $F_n(X)$ and its associated probability density $\rho_n(x)$ possess the following generic properties. (i) Since a new integration range can only open up by increasing x and the resulting contribution is necessarily non-negative, $F_n(x)$ is a monotonically increasing function of x, as indeed is expected. (ii) More unexpectedly, the slope of $F_n(x)$ with respect to x will be subjected to abrupt changes at the discrete set of x values corresponding to the successive crossing thresholds. At these values, it may increase or decrease, depending on the structure of the branches $f^{k\tau}(X_0)$ involved in the particular crossing configuration considered. (iii) Being the derivative of $F_n(x)$ with respect to x, the probability density $\rho_n(x)$ will possess discontinuities at the points of nondifferentiability of $F_n(x)$ and will, in general, be nonmonotonic.

Properties (ii) and (iii) are fundamentally different from those familiar from the statistical theory of extremes, where the corresponding distributions are smooth functions of x. In particular, the discontinuous nonmonotonic character of $\rho_n(x)$ complicates considerably the already delicate issue of prediction of extreme events [6]. We have thus identified some universal signatures of the deterministic character of the dynamics on the properties of extremes.

We now turn to the derivation of some more specific properties of $F_n(x)$ and $\rho_n(x)$ for three prototypical classes of dynamical systems. These systems have in common the property of ergodicity, guaranteeing the existence of a smooth invariant density. Furthermore, they are supposed to undergo a discrete time dynamics defining a one-dimensional mapping. Inasmuch as many key features of the dynamics of continuous time flows can be captured by reducing their evolution to such mappings on a suitable Poincaré surface of section [4,7], the results to be reported below should thus apply to large classes of physically relevant systems.

Uniform quasiperiodic motion.—The canonical form of the evolution law, whatever the detailed structure of the underlying system might be, is given by the twist map [4,7]

$$\phi_{n+1} = a + \phi_n \qquad \text{mod} 1, \quad (4)$$

where a is irrational. The invariant density $\rho(\phi)$ is uniform, $\rho(\phi) = 1$, and the associated cumulative distribution is $F(x) = x$. The expansion rates of the evolution law $f(\phi) = a + \phi$ and of its higher iterates are equal to unity, and there are neither fixed points nor periodic orbits. The discontinuities in the slopes of $F_n(x)$ thus arise solely from the first universal mechanism identified in our previous discussion, namely, the intersections between the different iterates of $f(\phi)$. According to Eq. (4), the left branches of these iterates cut the ordinate axis at the points $a, \ldots, na$ mod1, hereafter denoted for compactness as $\{a\}, \ldots \{na\}$, and the right branches cut the abscissa axis at the points $1 - \{a\}, \ldots 1 - \{na\}$. Now, a classic result of number theory [8] asserts that, given an irrational number a: (i) the set of $\{a\}, \ldots \{na\}$ arranged in ascending order partitions the unit interval into steps of at most three sizes, $\alpha = \{k_m a\}$, $\beta = 1 - \{k_M a\}$, and $\alpha + \beta$, where $\{k_m a\}$ and $\{k_M a\}$ are, respectively, the smallest and the largest of the values found in the set. (ii) There are $n + 1 - k_m$, $n + 1 - k_M$, $k_m + k_M - n - 1$ steps of length α, β, and $\alpha + \beta$, respectively. (iii) k_m, k_M, α, β, and n satisfy the relationships $k_m\beta + k_M\alpha = 1$, $n \geq \max(k_m, k_M)$, and $n \leq k_m + k_M - 1$.

It can be shown that these results have a direct bearing on the properties of recurrence times in quasiperiodic motion [9]. As we see presently, they also allow one, in conjunction with the symmetries entailed by the unit value of the slope of all iterates of $f(\phi)$, to evaluate $F_n(x)$ from Eq. (3). We illustrate the procedure for the contribution of

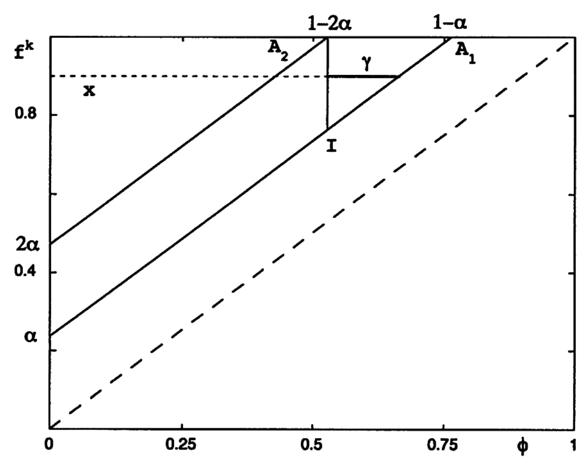

FIG. 1. Illustration of the origin of the contributions to the second term in Eq. (5), for $n = 4$ and $a = (\sqrt{5} - 1)/3$.

steps of size a to $F_n(x)$. Clearly (see Fig. 1), such steps are associated with the intersection of adjacent left branches of successive iterates of f^{km}. To fix ideas consider f^{km}, f^{km+1}, ..., which terminate at points A_1, A_2, ... of ordinates $1 - \alpha$ and $1 - 2\alpha$, ..., respectively. The intersection of two consecutive branches, say, f^{km} and f^{km+1}, are defined by the intersection of f^{km} and a vertical line drawn from point A_2 (point I in Fig. 1), and their ordinates are all equal to $1 - \alpha$. It follows that the theta functions in Eq. (3) will be satisfied provided that x lies above the threshold value $x = 1 - \alpha$. The gap opened up when x exceeds this threshold is the segment γ in Fig. 1. At the level of Eq. (3), it gives a contribution obtained by integrating over θ_0 between $1 - \alpha$ and x, yielding a value equal to $x - 1 + \alpha$. The argument can readily be extended to steps of sizes β and $\alpha + \beta$, leading to the following overall structure of $F_n(x)$:

$$\begin{aligned} F_n(x) = {} & (k_m + k_M - n)(x - 1 + \alpha + \beta) \\ & \times \theta(x - 1 + \alpha + \beta) + (n - k_m)(x - 1 + \alpha) \\ & \times \theta(x - 1 + \alpha) + (n - k_M)(x - 1 + \beta) \\ & \times \theta(x - 1 + \beta). \end{aligned} \tag{5}$$

We conclude that, generically, $F_n(x)$ undergoes three slope changes at the x values $1 - \alpha - \beta$, $1 - \alpha$, and $1 - \beta$. Its rightmost part starts at $x = 1 - \max(\alpha, \beta)$ and has a slope given by $k_m + k_M - n + n - k_m + n - k_M = n$. Its first nontrivial leftmost part starts at $x = 1 - \alpha - \beta$ and has a slope given by $k_m + k_M - n$, which may, but does not have to, be equal to one. These conclusions are fully confirmed by the results of direct numerical simulation based on Eq. (4), as depicted in Fig. 2.

Fully developed chaotic maps in the interval.— Dynamical systems exhibiting this kind of behavior are at the core of classical chaos theory. They do not derive from a canonical form as in Eq. (4). Still, they share some common features such as to possess a mean expansion rate larger than 1 and an exponentially large number of unstable periodic trajectories [10]. In view of the comments following Eq. (3), these properties will show up through the presence of an exponentially large number of points in which $F_n(x)$ will change slope and an exponentially large number of plateaus of the associated probability density $\rho_n(x)$. One may refer to this latter peculiar property as a "generalized devil's staircase." As a corollary, the first smooth segment of $F_n(x)$ will have a support of $O(1)$ and the last one an exponentially small support, delimited by the rightmost fixed point of the iterate $f^{(n-1)}$ and the right boundary b of the interval. Since $F_n(x)$ is monotonic and $F_n(b) = 1$, the slopes will be exponentially small in the first segments and will gradually increase as x approaches b. Again, these properties differ markedly from the classical statistical theory of extremes. As an illustration, Fig. 3 depicts the functions $F_{20}(x)$ and $\rho_{20}(x)$ as deduced by direct numerical simulation of the tent map [4], $f(X) = 1 - |1 - 2X|$, $0 \leq X \leq 1$. The results confirm entirely the

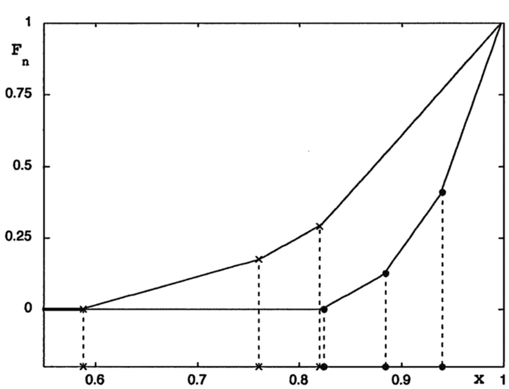

FIG. 2. Cumulative probability distribution $F_n(x)$ for the twist map, Eq. (4) with $a = (\sqrt{5} - 1)/3$. The upper curve corresponds to $n = 4$ and the lower to $n = 10$. Notice that in both cases the number of slope changes in $F_n(x)$ is equal to 3 as predicted by Eq. (5), whose positions are indicated in the figure by vertical dashed lines.

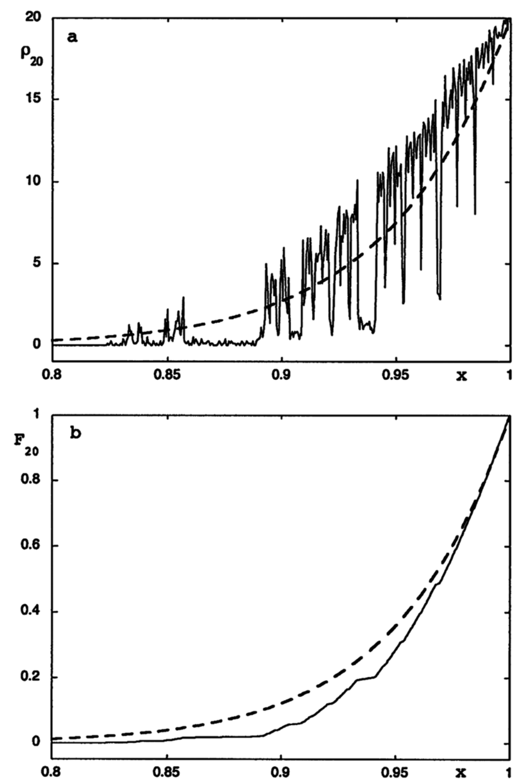

FIG. 3. (a) Probability density and (b) cumulative probability distribution for the tent map as obtained numerically using 10^6 realizations. Dashed curves represent the prediction of the classical statistical theory of extremes. The irregular succession of plateaus in $\rho_{20}(x)$ and the increase of the slope of $F_{20}(x)$ in the final part of the interval are in full agreement with the general theory. The irregularity increases rapidly with the window (contrary to the quasiperiodic case), and there is no saturation and convergence to a smooth behavior in the limit of infinite window.

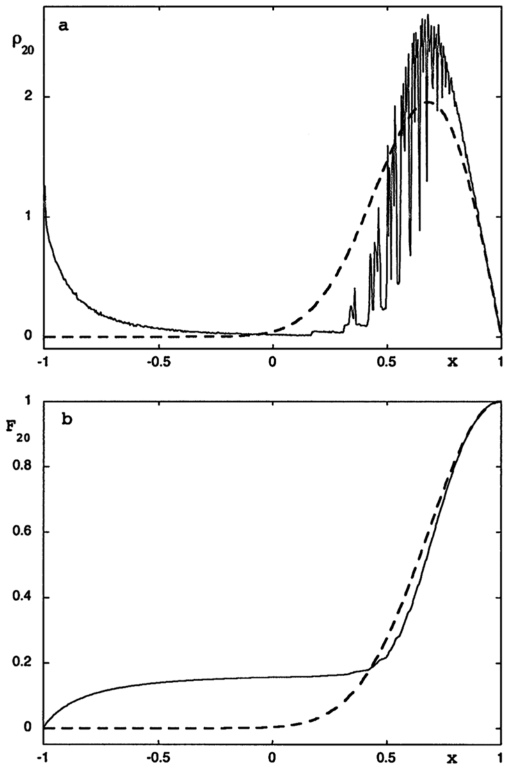

FIG. 4. As in Fig. 3 but for the cusp map. The irregularities pointed out in connection with Fig. 3 subsist, the main new point being the presence of a more appreciable probability mass in the left part of the interval.

theoretical predictions. The simplicity of this dynamical system allows one to establish a further result of interest, namely, that the probability mass in the interval $(1 - 1/n, 1)$ is of $O(1)$, which also happens to be the case in the classical statistical theory.

Intermittent chaotic maps in the interval.—Intermittent systems share the property to possess mean expansion rates close to unity in some regions of phase space or, in their implementation as one-dimensional maps, a slope of $f(X)$ versus X close to unity. We take, without loss of generality, this property to occur in the leftmost boundary a. A widely used mapping in chaos theory satisfying this condition is [7]

$$f(X) \approx (X - a) + u|X - a|^z + \varepsilon, \qquad a \leq X \leq b, \quad (6)$$

where $z > 1$ and ε measures the distance from strict tangency. As $\varepsilon \to 0$ successive iterates $f^k(X)$ will follow the $f(X) = X$ axis closer and closer and are thus bound to become increasingly steep at their respective reinjection points where $f^k(X) = b$. As a result, the positions of these points [and hence of the (unstable) fixed points other than from $X = a$, too, whose number is still exponentially large] will move much more slowly towards a and b compared to the fully chaotic case. Two new qualitative properties of $F_n(x)$ can be expected on these grounds: The probability mass borne in the first smooth segment of this function near $X = a$ and the length of the last smooth segment near $X = b$ will no longer be exponentially small. This is fully confirmed by direct numerical simulation of Eq. (6) for the symmetric cusp map [3], $f(X) = 1 - 2|X|^{1/2}$, $-1 \leq X \leq 1$, as seen in Fig. 4. Using the explicit form of $f(X)$, one can check straightforwardly that $F_n(x) \approx 1 + x$ as $x \to -1$, $F_n(0) \approx n^{-1}$, a final segment of $F_n(x)$ of width $O(n^{-1})$, and $F_n(x) \approx 1 - n(1 - x)^2/4$ as $x \to 1$.

We finally comment on the limiting behavior of the extreme value distributions. Since nondifferentiability of $F_n(x)$ holds for any window value, it is bound to subsist at large n. No simple limiting behavior is thus to be expected. In particular, for fully developed chaos, the first point of nondifferentiability of $F_n(x)$ will necessarily lie at a finite distance from the boundaries of the interval of variation of x. The situation may be more clear-cut in uniform quasi-periodic motion, where, under certain conditions [8], α and β in Eq. (5) tend to zero as $n \to \infty$, entailing that non-differentiability is squeezed near the upper boundary. This question deserves further investigation. Future investigations in this area should aim at analyzing the statistics of extremes in more detailed models describing concrete chemical, fluid mechanical, or geophysical processes. Furthermore, the experimental data available on extreme values should be reconsidered in the light of the results reported in this work.

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[1] E. W. Montroll and W. W. Badger, *Introduction to Quantitative Aspects of Social Phenomena* (Gordon and Breach, New York, 1974).

[2] P. Embrechts, C. Klüppelberg, and T. Mikosch, *Modelling Extremal Events* (Springer, Berlin, 1999).

[3] M. R. Leadbetter and H. Rootzen, Ann. Probab. **16**, 431 (1988).

[4] G. Nicolis, *Introduction to Nonlinear Science* (Cambridge University Press, Cambridge, England, 1995).

[5] A. Lasota and M. Mackey, *Probabilistic Properties of Deterministic Systems* (Cambridge University Press, Cambridge, England, 1985).

[6] V. Balakrishnan, C. Nicolis, and G. Nicolis, J. Stat. Phys. **80**, 307 (1995).

[7] H. G. Schuster, *Deterministic Chaos* (VCH Verlag, Weinheim, 1988).

[8] N. B. Slater, Proc. Cambridge Philos. Soc. **63**, 1115 (1967).

[9] M. Theunissen, C. Nicolis, and G. Nicolis, J. Stat. Phys. **94**, 437 (1999).

[10] R. Devaney, *Chaotic Dynamical Systems* (Addison-Wesley, Redwood City, CA, 1989).