Spectral Theory of Metastability and Extinction in Birth-Death Systems

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(Received 21 August 2006; published 17 November 2006)

We suggest a general spectral method for calculating the statistics of multistep birth-death processes and chemical reactions of the type $mA \to nA$ (m and n are positive integers) which possess an absorbing state. The method employs the generating function formalism in conjunction with the Sturm-Liouville theory of linear differential operators. It yields accurate results for the extinction statistics and for the quasistationary probability distribution, including large deviations, of the metastable state. The power of the method is demonstrated on the example of binary annihilation and triple branching $2A \to \emptyset$, $A \to 3A$, representative of the rather general class of dissociation-recombination reactions.

DOI: 10.1103/PhysRevLett.97.200602 PACS numbers: 05.40.−a, 02.50.Ey, 82.20.−w, 87.23.Cc

Since the pioneering works of Delbrück [1], Bartholomay [2] and McQuarrie [3], kinetics of systems of birth-death type, containing a large but finite number of agents (such as molecules, bacteria, cells, animals, or even humans), have attracted much attention in different areas of science and become a paradigm of theory of stochastic processes [4,5]. Birth-death models are extensively discussed in chemistry, astrochemistry, epidemiology, population biology, cell biochemistry, etc. They are also well known in nonequilibrium physics and can be viewed in the context of reaction-limited kinetics on a lattice, as opposed to more extensively studied diffusion-limited kinetics [6]. While the behavior of the average number of particles in such systems may be describable, at not too long times, by (mean-field) *rate equations*, fluctuations may lead to important quantitative or even qualitative differences. This necessitates using the more general *master equation* which deals with the probability of having a certain number of particles of each type at time t. The master equation is rarely solvable analytically, and various approximations, often uncontrolled, are in use [4,5], such as the Fokker-Planck equation which may suffice unless one has to deal with large deviations or extinction phenomena [7–9]. Not much is known beyond the Fokker-Planck description, though in particular cases, the statistics were determined by using approximations in the master equation [9–15] or, alternatively, by introducing a generating function [3–5], see below, and developing different approximations in the equation describing its evolution [8,16,17].

In this Letter, we advance the generating function technique by marrying it with the Sturm-Liouville theory of linear differential operators. This yields a general and robust *spectral* formalism, capable of providing accurate, and often analytical, results for extreme statistics in a variety of (not necessarily single-step) birth-death systems and chemical reactions. We demonstrate the power of our method by a simple reaction of binary annihilation and triple branching. An example of such a reaction is recombination of two atoms A and dissociation of the molecule A_2: $A + A \to A_2$, and $A_2 + A \to 3A$, assuming that the A_2 molecules are always at hand [18]. For H or N atoms, this reaction occurs at high temperatures [19]. We calculate the extinction probability as a function of time, the mean time to extinction, and the complete quasistationary probability distribution of the long-lived metastable state, intrinsic to this problem.

Rate equation, master equation, generating function, and spectral theory.— Consider the binary annihilation and triple branching reactions $2A \xrightarrow{\mu} \emptyset$ and $A \xrightarrow{\lambda} 3A$, where $\mu, \lambda > 0$ are rate constants. The rate equation (or the mean-field theory) of this simple system, $dn/dt = 2\lambda n - \mu n^2$, describes a nontrivial attracting steady state $n_s = 2\Omega$, where $\Omega = \lambda/\mu$. Fluctuations, caused by discreteness of particles, invalidate this mean-field result owing to the existence of the absorbing state $n = 0$: a state from which there is a zero probability of exiting. At $\Omega \gg 1$, however, a long-lived (and therefore quasistationary) fluctuating *metastable* state is observed, once the initial population is not too sparse. The statistics of the quasistationary state and of the extinction times are the subjects of our interest here.

To account for discreteness of particles, we assume that the evolution of the probability $P_n(t)$ to find n particles at time t is described, for $n > 1$, by the master equation

$$\frac{d}{dt}P_n(t) = \frac{\mu}{2}[(n+2)(n+1)P_{n+2}(t) - n(n-1)P_n(t)] + \lambda[(n-2)P_{n-2}(t) - nP_n(t)]. \quad (1)$$

Let us introduce the generating function [3–5]

$$G(x,t) = \sum_{n=0}^{\infty} x^n P_n(t), \quad (2)$$

where x is an auxiliary variable. $G(x,t)$ encodes all the probabilities:

$$P_n(t) = \frac{1}{n!}\frac{\partial^n G(x,t)}{\partial x^n}\bigg|_{x=0}. \quad (3)$$

Obviously, $G(x=1,t) = 1$. Eqs. (1) and (2) yield a partial differential equation (PDE) for $G(x, t)$:

$$\frac{\partial G}{\partial t} = \frac{\mu}{2}(1 - x^2)\frac{\partial^2 G}{\partial x^2} + \lambda x(x^2 - 1)\frac{\partial G}{\partial x}. \quad (4)$$

As the reaction we are dealing with conserves parity, $G(x, t)$ can be written as

$$G(x, t) = c_1 G_{\rm even}(x, t) + c_2 G_{\rm odd}(x, t), \quad (5)$$

where $c_1 = \sum_0^\infty P_{2m}(t = 0)$ and $c_2 = 1 - c_1$. Therefore, $G_{\rm even}(x = \pm 1, t) = 1$ and $G_{\rm odd}(x = \pm 1, t) = \pm 1$. The steady state solution of Eq. (4) is

$$G_{\rm st}(x, t) = c_1 + c_2 \frac{{\rm erfi}(\sqrt{\Omega} x)}{{\rm erfi}(\sqrt{\Omega})}, \quad (6)$$

where ${\rm erfi}(x) = (2/\sqrt{\pi}) \int_0^x e^{t^2} dt$. Let the number of particles at $t = 0$ be *even*: $n_0 = 2k_0$, where k_0 is integer. In this case, the parity conservation yields $c_1 = 1$ and $c_2 = 0$, so $G_{\rm st}(x) = 1$, and the only true steady state is the empty state: $P_0 = 1$, while the rest of P_n are zero [20].

To see how the population of $n_0 = 2k_0$ particles at $t = 0$ becomes extinct, we introduce a new function $g(x, t) = G(x, t) - G_{\rm st}(x) = G(x, t) - 1$ which obeys Eq. (4) with homogenous boundary conditions $g(x = \pm 1, t) = 0$. Substituting $g(x, t) = e^{-\gamma t}\varphi(x)$, we obtain

$$(1 - x^2)\varphi''(x) + 2\Omega x(x^2 - 1)\varphi'(x) + 2E\varphi(x) = 0, \quad (7)$$

where $E = \gamma/\mu$. Rewriting this ordinary differential equation in a self-adjoint form,

$$[\varphi'(x)\exp(-\Omega x^2)]' + Ew(x)\varphi(x) = 0, \quad (8)$$

with the weight function $w(x) = 2\exp(-\Omega x^2)(1 - x^2)^{-1}$, we arrive at a standard eigenvalue problem of the Sturm-Liouville theory [21]. Once we have found the complete set of orthogonal eigenfunctions $\varphi_k(x)$ (which are all even), and the real eigenvalues E_k, $k = 1, 2, \ldots$, we can solve the time-dependent problem:

$$G(x, t) = 1 + \sum_{k=1}^{\infty} a_k \varphi_k(x) e^{-\mu E_k t}, \quad (9)$$

where

$$a_k = \frac{\int_0^1 [G(x, t = 0) - 1]\varphi_k(x) w(x) dx}{\int_0^1 \varphi_k^2(x) w(x) dx}, \quad (10)$$

and $G(x, t = 0) = x^{2k_0}$.

As all $E_k > 0$, Eq. (9) describes *decay* of initially populated states $k = 1, 2, \ldots$, and the system approaches the empty state $G(x, t \to \infty) = 1$. We are interested in the case of $\Omega \gg 1$, where the metastable state is expected to be long-lived. Elgart and Kamenev [22] showed that the eigenvalues $E_2, E_3, \ldots$ scale like $\mathcal{O}(\Omega) \gg 1$. In contrast to these, the "ground-state" eigenvalue E_1 is exponentially small, as will be proved *a posteriori*. We will be interested in sufficiently long times $\mu\Omega t = \lambda t \gg 1$, when the contribution from the "excited" states to $G(x, t)$ becomes negligible, and we can write

$$G(x, t) = 1 + a_1 \varphi_1(x) e^{-\mu E_1 t}. \quad (11)$$

Let us proceed to the ground-state calculations.

Ground-state calculations.—As $\varphi_1(x) \equiv \varphi(x)$ is an even function, it suffices to consider the interval $0 \le x \le 1$. We will employ the strong inequality $\Omega \gg 1$ and find the (very small) eigenvalue E_1 and the corresponding eigenfunction of Eq. (7) by a matched asymptotic expansion, see, e.g., Ref. [23]. In most of the region $0 \le x < 1$ (the bulk), we can treat the last term in Eq. (7) perturbatively. In the zero order, we put $E_1 = 0$ and arrive at the *steady state* equation $\varphi_b''(x) - 2\Omega x \varphi_b'(x) = 0$, whose even solution is $\varphi_b^{(0)} = 1$. Now, we put $\varphi_b(x) = 1 + \delta\varphi_b(x)$, where $\delta\varphi_b(x) \ll 1$, and obtain in the first order

$$\delta\varphi_b''(x) - 2\Omega x \delta\varphi_b'(x) = -2E_1(1 - x^2)^{-1}. \quad (12)$$

The solution for $\varphi_b(x)$ takes the form

$$\varphi_b(x) = 1 - 2E_1 \int_0^x e^{\Omega s^2} ds \int_0^s \frac{e^{-\Omega r^2}}{1 - r^2} dr. \quad (13)$$

As $\Omega \gg 1$, we can omit the r^2 term in the denominator of the inner integral in Eq. (13) (this omission is illegitimate too close to $x = 1$, but the bulk solution is invalid there anyway, see below). We obtain

$$\varphi_b(x) \simeq 1 - 2E_1 \int_0^x e^{\Omega s^2} ds \int_0^s e^{-\Omega r^2} dr$$
$$= 1 - E_1 x^2 {}_2F_2\left(1, 1; \frac{3}{2}, 2; \Omega x^2\right), \quad (14)$$

where ${}_2F_2(a_1, a_2; b_1, b_2; x)$ is the generalized hypergeometric function [24], while E_1 is yet unknown. It is easy to check that the bulk solution is valid $[\delta\varphi_b(x) \ll 1]$ as long as $1 - x \gg 1/\Omega$.

In the boundary layer $1 - x \ll 1$, we can again disregard, at $\Omega \gg 1$, the (exponentially small) last term in Eq. (7). The resulting equation is again $\varphi_l''(x) - 2\Omega x \varphi_l'(x) = 0$. Its nontrivial solution, obeying the boundary condition at $x = 1$, is

$$\varphi_l(x) = C \int_x^1 e^{\Omega s^2} ds \simeq C \frac{e^{\Omega}}{2\Omega}[1 - e^{-\Omega(1-x^2)}], \quad (15)$$

where C is a yet unknown constant.

Now we demand that, in the common region $1/\Omega \ll 1 - x \ll 1$, the proper asymptote of the bulk solution (14), obtained by moving to infinity the upper limit in the inner integral of Eq. (14),

$$\varphi_b(x) \simeq 1 - \frac{\sqrt{\pi} E_1}{\sqrt{\Omega}} \int_0^x e^{\Omega s^2} ds \simeq 1 - \frac{\sqrt{\pi} E_1}{2\Omega^{3/2}} e^{\Omega x^2}, \quad (16)$$

coincides with the boundary layer solution (15). Eqs. (15) and (16) yield

$$E_1 = \frac{2\Omega^{3/2}}{\sqrt{\pi}} e^{-\Omega} \quad \text{and} \quad C = 2\Omega e^{-\Omega}. \tag{17}$$

As expected, the lowest eigenvalue E_1 is exponentially small in Ω. The respective eigenfunction is

$$\varphi(x) \simeq \begin{cases} \varphi_b(x) = 1 - E_1 x^2 {}_2F_2(1, 1; \frac{3}{2}, 2; \Omega x^2) & \text{for } 1 - x^2 \gg 1/\Omega, \\ \varphi_l(x) = 1 - e^{-\Omega(1-x^2)} & \text{for } 1 - x^2 \ll 1. \end{cases} \tag{18}$$

Now we use Eq. (10) to calculate the coefficient a_1 entering Eq. (11). While evaluating the integrals, we notice that the main contributions come from the bulk region $1 - x^2 \gg 1/\Omega$, and it suffices to take the eigenfunction $\varphi_b(x)$ in the zeroth order, that is $\varphi_b^{(0)}(x) \simeq 1$. Evaluating the integral in the nominator of Eq. (10), we notice that the term x^{2k_0} under the integral is negligible compared to 1. As a result, the integrals in the nominator and denominator become identical up to a minus sign. Therefore, $a_1 \simeq -1$, which completes our solution (11).

Statistics of the quasistationary state.—We start this part with calculating the average number of particles $\bar{n}$ and the standard deviation σ at intermediate times $\Omega^{-1} \ll \mu t \ll E_1^{-1}$. Using Eq. (11), we obtain

$$\bar{n} = \partial_x G|_{x=1} = 2\Omega, \tag{19}$$

which coincides with the mean-field result. Furthermore,

$$\sigma^2 = \bar{n^2} - \bar{n}^2 = [\partial_{xx}^2 G + \partial_x G - (\partial_x G)^2]|_{x=1} = 4\Omega, \tag{20}$$

where we have used for $\varphi(x)$ its boundary layer asymptote $\varphi_l(x)$ from Eq. (18). One can see that, at intermediate times $\Omega^{-1} \ll \mu t \ll E_1^{-1}$, the system stays in the (weakly fluctuating) quasistationary state. What is the complete probability distribution $P_n(t)$ of the quasistationary state at these times? For $n = 0$, we obtain

$$P_0(t) = G(x = 0, t) = 1 - e^{-\mu E_1 t} \tag{21}$$

which, at $\mu E_1 t \ll 1$, is very small. For (even) nonzero values of n, Eqs. (3) and (11) yield

$$P_n(t) = \frac{2E_1(4\Omega)^{n/2-1}(n/2 - 1)!}{n!} e^{-\mu E_1 t}. \tag{22}$$

For $n \gg 1$, we can use Stirling's formula and obtain

$$P_n(t) \simeq \frac{E_1}{\sqrt{2n\Omega}} e^{(n/2)[1+\ln(2\Omega/n)] - \mu E_1 t}. \tag{23}$$

Notably, *all* of the probabilities $P_n(t)$ ($n > 0$) decay with time, while $P_0(t)$ grows. One can check that the most probable state coincides with $\bar{n} = 2\Omega$. In the vicinity of $n = \bar{n}$, $P_n(t)$ from Eq. (23) can be approximated by a normal distribution with the mean $\bar{n}$ and standard deviation σ, given by Eqs. (19) and (20), respectively. The tails of the true distribution, however, are strongly non-Gaussian. A comparison between our analytic result (22), the large-n approximation (23), and the normal distribution is shown in Fig. 1. One can see that Eq. (23) is very accurate, whereas the Gaussian approximation strongly overpopulates the low-n tail and underpopulates the high-n tail.

Figure 2 compares our analytic result (22) with a numerical solution of the (truncated) master Eq. (1) with $(d/dt)P_n(t)$ replaced by zeros and $P_0 = 0$. The two curves are almost indistinguishable for $\Omega = 10$. In fact, good agreement is observed already for $\Omega = \mathcal{O}(1)$, and it rapidly improves further as Ω increases.

Extinction time statistics.—The quantity $P_0(t)$ is the probability of extinction at time t. The extinction probability density is $p(t) = dP_0(t)/dt$. Using Eq. (21), we obtain an exponential distribution:

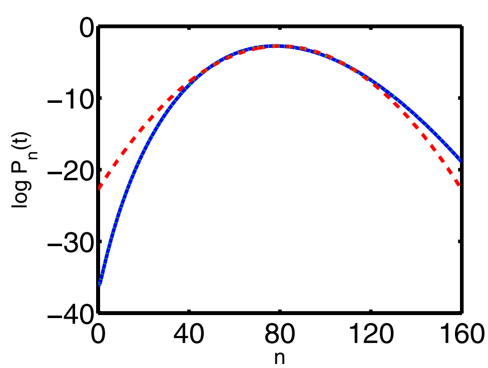

FIG. 1 (color online). The natural logarithms of the probability distribution (22) (the solid line), of its large-n asymptotics (23) (the dotted line), and of the normal distribution with $\bar{n}$ and σ from Eqs. (19) and (20) (the dashed line), for $\Omega = 40$ and $\mu E_1 t \ll 1$.

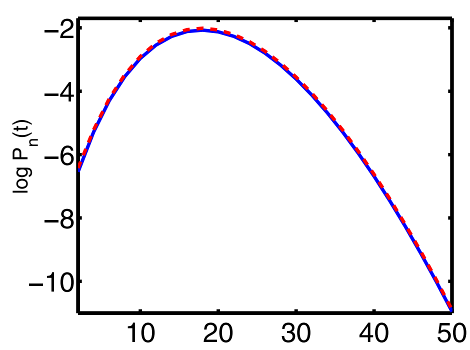

FIG. 2 (color online). The dashed line: the natural logarithm of the probability distribution (22) at $\mu E_1 t \ll 1$. The solid line: a numerical solution of the master Eq. (1), see text, for $\Omega = 10$.

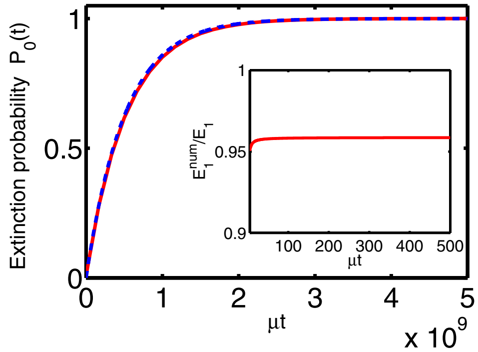

FIG. 3 (color online). Extinction probability $P_0(t)$ from Eq. (21) (the dashed line) and from a numerical solution of Eq. (4) (the solid line) for $\Omega = 25$ and $n_0 = 2k_0 = 100$. The inset shows, at times $\mu t \gg 1/\Omega$, the ratio of the numerical ground-state eigenvalue $E_1^{\rm num} = -\log[1 - G(0, t)]/\mu$ and the analytical one, given by Eq. (17). The deviation is less than 4%; that is within error $\mathcal{O}(1/\Omega)$.

$$p(t) \simeq \mu E_1 e^{-\mu E_1 t} \quad \text{at} \quad \lambda t \gg 1. \tag{24}$$

The average time to extinction, $\bar{\tau} = \int_0^\infty tp(t)dt \simeq (\mu E_1)^{-1}$, is exponentially large, at $\Omega \gg 1$, as expected. Figure 3 compares the analytical result (21) for $P_0(t)$ with $G(0, t)$ found by solving Eq. (4) numerically with the boundary conditions $G(\pm 1, t) = 1$ and the initial condition $G(x, t = 0) = x^{2k_0}$. The inset compares the analytical and numerical ground-state eigenvalues, and good agreement is observed.

Final comments.—The spectral formalism yields accurate extinction time statistics and the complete quasistationary probability distribution of the metastable state for a wide class of birth-death processes which possess an absorbing state and are describable by a master equation. In this formalism, the problem of computing these statistics is reduced to a problem (familiar to every physicist) of finding a ground-state eigenfunction and eigenvalue of a linear differential operator. We have demonstrated the formalism by an example of binary annihilation and triple branching, but the formalism is general and can be used for a variety of kinetics. In most interesting cases of *long-lived* metastable states, a large parameter (the average number of particles in the metastable state) is always present in the problem. This paves the way to a perturbative treatment, like in the example we have considered.

The spectral formalism should be also efficient when the absorbing state is at infinity, rather than at zero. For systems of this type, the rate equation yields a stable nonempty steady state, but an account of fluctuations brings about an unlimited population growth, see, e.g., Ref. [8]. Finally, the use of spectral formalism is not at all limited to systems possessing an absorbing state [17].

We acknowledge useful discussions with Alex Kamenev and Vlad Elgart and thank Len Sander and Uri Asaf for advice. The work was supported by the Israel Science Foundation (Grant No. 107/05) and by the German-Israel Foundation for Scientific Research and Development (Grant No. I-795-166.10/2003).

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