Holographic Probabilities in Eternal Inflation

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In the global description of eternal inflation, probabilities for vacua are notoriously ambiguous. The local point of view is preferred by holography and naturally picks out a simple probability measure. It is insensitive to large expansion factors or lifetimes and so resolves a recently noted paradox. Any cosmological measure must be complemented with the probability for observers to emerge in a given vacuum. In lieu of anthropic criteria, I propose to estimate this by the entropy that can be produced in a local patch. This allows for prior-free predictions.

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The evidence for nonvanishing vacuum energy suggests that fundamental theory has an enormous number of long-lived, metastable vacua [1]. Happily, string theory appears to satisfy this criterion [2,3]. Then many low-energy parameters will not be determined uniquely but statistically. To make pre- or postdictions, one must survey representative samples of the landscape of vacua [4,5]. But this is not enough. Cosmological dynamics may favor the production of some vacua and suppress others. The predictivity of fundamental theory hinges on a quantitative understanding of this effect.

The vacua with a positive cosmological constant trap the Universe in eternal inflation. They decay only locally, by producing bubbles of new vacua. Thus, every vacuum in the landscape will be realized an infinite number of times in different, causally disconnected regions. Each bubble expands to become an infinite open Universe embedded in the global spacetime.

To regulate these infinities, one might compare the prevalence of different vacua at a finite time and then take a late time limit. But this task is plagued by ambiguities [6]. Should vacuum i be weighted by the number of i bubbles or by their volume? Worse, both quantities depend on the choice of time variable, and there is no preferred time slicing in eternal inflation. A number of interesting probability measures have been proposed, most recently in Refs. [7–9]. That they give different answers illustrates the intricacy of the problem. One can imagine other prescriptions, and practically any answer can be obtained by devising a suitable time slicing.

Here I will develop a probability measure by appealing only to a single causally connected region, or causal diamond [10]. This is called the local, or causal, or holographic point of view. My approach will encounter none of the ambiguities listed above. Moreover, there are independent reasons to embrace this viewpoint: From the quantum properties of black holes, we have learned that the simultaneous description of two causally disconnected regions leads to paradoxes, which are resolved if we stick to describing only what any one observer can measure [11]. In fact, the situation in eternal inflation is worse than for black holes. An observer outside a black hole can compute the interior geometry from initial conditions, but an observer in eternal inflation cannot predict when and where bubbles will form and so cannot distinguish between macroscopically distinct global metrics [12]. Thus, the local observer cannot even construct a global geometry about whose slicing one could argue.

I consider only a single worldline, so the task breaks up into two parts: (i) *Prior probability*—How likely is it for the worldline to enter vacuum i? (ii) *Weighting*—What is the probability that observers will emerge in vacuum i? On the latter issue, I will find that the causal viewpoint permits the elimination of anthropic selection criteria—which are hard to specify for widely varying low-energy theories—in favor of prior-free thermodynamic conditions for the emergence of complex phenomena such as observers.

Prior probability.—Consider a landscape with vacua i. These should include metastable vacua, which eventually decay into other vacua. There may also be "terminal" vacua, which do not decay (typically, the vacua with a nonpositive cosmological constant). If so, the landscape is called terminal; otherwise, it is termed "cyclic." (There is empirical evidence that the landscape is terminal [13,14]. Moreover, the landscape of string theory is terminal.) There will be no need to restrict to the terminal case here, but I will assume that the landscape is connected: Every vacuum can be reached from any metastable vacuum by some sequence of decays.

How likely is it for the worldline to enter a given vacuum on its way through the landscape? (The question is *not* how much time the worldline is likely to spend in i. Complex phenomena such as observers arise between bubble formation and thermalization. Typical lifetimes of vacua are exponentially longer than this out-of-equilibrium period, so their inclusion at this point would lead to huge correction factors in the final section. The length of the prehistory is equally irrelevant. Never mind how long the worldline lingers in a metastable vacuum a; the question is which vacuum it enters next.) Let κ_{ij} be the probability per unit proper time for a geodesic worldline in vacuum j to enter vacuum i. Normalize each column of κ to sum to 1,

$\mu_{ia} = \kappa_{ia}/\sum_j \kappa_{ja}$, except for columns corresponding to terminal vacua, which vanish. The matrix μ describes the *relative* probability to decay from a to i.

Now draw a root node labeled o, corresponding to the initial vacuum the worldline starts out in. For each vacuum i that o can decay into, draw a branch connecting o with a new node labeled with the new vacuum i. Next to each branch, write the relative probability μ_{branch} for this decay channel (in this case, $\mu_{\text{branch}} = \mu_{io}$). Then repeat this procedure for each metastable new vacuum. This will generate a tree.

Next, compute a raw (i.e., unnormalized) probability for each vacuum in the landscape. For each path from the root node to the vacuum in question, multiply the branch probabilities; then sum up the results:

$$P_i = \sum_{\substack{\text{all nodes}\\ \text{labeled } i}} \prod_{\substack{\text{the branches connecting}\\ \text{the root to the node}}} \mu_{\text{branch}}. \qquad (1)$$

The normalized probability for a worldline to pass through vacuum i is $p_i = P_i/\sum_j P_j$.

For a simple example, consider a landscape with two metastable vacua A and B, and a terminal vacuum Z, as shown in Fig. 1. In this model, A can only decay to B ($\eta_{BA} = 1$). The vacuum B decays to Z with probability $\eta_{ZB} = 1 - \epsilon$ or back up to A with probability $\eta_{AB} = \epsilon$.

First, suppose that the initial vacuum is A. From the associated tree (Fig. 1, left), one sees that there are infinitely many paths leading into each vacuum. For vacuum A, the paths are ABA, $ABABA$, …, giving a raw probability $P_A = \epsilon + \epsilon^2 + \ldots = \epsilon/(1-\epsilon)$. For vacuum B, the paths are AB, $ABAB$, etc., and the vacuum Z arises from paths ABZ, $ABABZ$, etc. After normalization, one obtains $p_A = \epsilon/2$, $p_B = 1/2$, $p_Z = (1-\epsilon)/2$.

Now suppose that the initial vacuum is B (Fig. 1, right). One finds $p_A = p_B = \epsilon/(1+\epsilon)$, $p_Z = (1-\epsilon)/(1+\epsilon)$. As one would expect for a single worldline, the probability to pass through a given vacuum can depend on the initial vacuum. (Some of the extant proposals depend strongly on initial conditions [7], others more weakly [8]. But this is hardly a criterion for evaluating them, since we have neither observational nor theoretical grounds to demand *a priori* that the result of this particular dynamical process be insensitive to the starting point. The initial probability distribution is an independent theoretical problem; see Ref. [15] for an opinionated discussion of some proposals.) It is interesting to take note of the limiting values of the above probabilities as $\epsilon \to 0$ or $\epsilon \to 1$. These are physically the most relevant cases, because the rates for different decay channels generically differ by exponentially large factors.

The formulation so far is not quite perfect, since the raw probabilities in Eq. (1) need not be finite. It is useful to think of the tree in terms of a conserved probability current, which enters at the root (the source) and flows up, ending up exclusively in terminal vacua (the sinks). It follows that the total *raw* probability for all terminal vacua is unity: $\sum_z P_z = 1$, where the sum runs over terminal vacua. The connectedness of the landscape then implies that all vacua have finite raw probability, if there is at least one terminal vacuum.

The *pruned tree* is constructed like the full tree, except that one terminates the tree wherever it returns to the initial vacuum o (Fig. 2). One can compute raw probabilities by applying Eq. (1) to the pruned tree. Now the conservation of the probability current implies that $\mathcal{P}_o + \sum_z \mathcal{P}_z = 1$, so that all raw probabilities computed from the pruned tree are finite. Because o is now effectively treated like a terminal vacuum, this conclusion applies independently of the presence of actual terminal vacua.

The full tree can be reconstructed from the pruned tree by joining a copy of the pruned tree at the root to every final node labeled o in the original pruned tree and iterating (see Fig. 2). This means that the raw probabilities of the

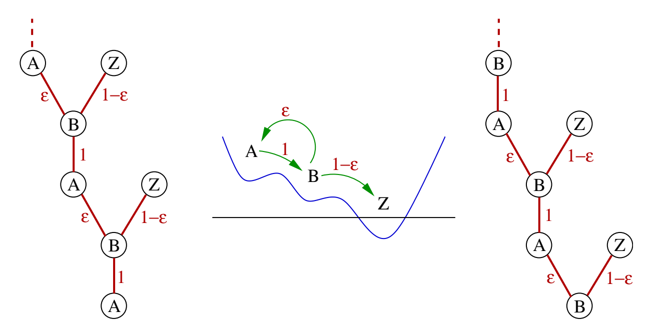

FIG. 1 (color online). A landscape with two metastable vacua and one terminal vacuum. The tree on the left corresponds to a worldline starting in vacuum A (the initial vacuum, or root). The tree on the right starts with vacuum B. The unnormalized probability for vacuum i is obtained by computing the probability for each path leading up from the root to i (the product of the numbers along the path) and summing over all paths.

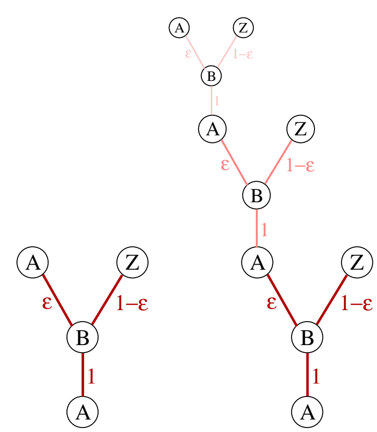

FIG. 2 (color online). Probabilities are easier to compute from the pruned tree, shown left for the A, B, and Z models, with initial vacuum A. One reads off readily that $\mathcal{P}_A = \epsilon$, $\mathcal{P}_B = 1$, and $\mathcal{P}_Z = 1 - \epsilon$, which need only be normalized. Right: The full tree can be recovered by iterating the pruned tree. Each iteration changes all raw probabilities by the same factor, leaving the normalized probabilities invariant.

full tree P_i will be given by $P_i = \mathcal{P}_i(1 + \mathcal{P}_o + \mathcal{P}_o^2 + \ldots) = \mathcal{P}_i/\sum_z \mathcal{P}_z$. It follows that the raw probabilities computed from the full tree P_i converge if and only if the landscape is terminal. If they do converge, then the full and the pruned trees yield the same normalized probabilities. Thus, the pruned tree yields the most general prescription.

For example, consider a cyclic landscape with three metastable vacua, as shown in Fig. 3. For simplicity, assume that A and C cannot decay into each other directly but only through B. The pruned tree depends on the initial vacuum, but the normalized probabilities do not: $p_A = \epsilon/2$, $p_B = 1/2$, $p_C = (1-\epsilon)/2$. Below, it will be shown that p_i is always independent of the initial condition in a cyclic landscape.

Matrix formulation.—It is intuitive to compute the prior probabilities from tree graphs, but it is also useful to reformulate the result as a matrix equation. The initial probability vector $\mathbf{P}^{(0)}$ satisfies $P_j^{(0)} = 1$ for $j = o$ and $P_j^{(0)} = 0$ otherwise. [The result, Eq. (2) below, will naturally incorporate more general initial probability distributions $\mathbf{P}^{(0)}$.] Let us consider the partial probability $P_i^{(\alpha)}$ to reach vacuum i from o after exactly α steps.

On a full tree, the partial probabilities obey $\mathbf{P}^{(\alpha)} = \eta \mathbf{P}^{(\alpha-1)}$. The raw probability is the sum of partial probabilities: $\mathbf{P} = \sum_{\alpha=1}^{\infty} \mathbf{P}^{(\alpha)}$. These two equations imply that the raw probability obeys the matrix equation

$$(1 - \eta)\mathbf{P} = \eta \mathbf{P}^{(0)}. \tag{2}$$

To be consistent with the results above, this equation should have a solution if and only if the landscape is terminal. Let us prove this. Suppose that there is no solution. Then $(1-\eta)$ cannot be invertible, and η must have an eigenvalue 1 with eigenvector $\tilde{\mathbf{P}}$, satisfying $\tilde{\mathbf{P}} = \eta\tilde{\mathbf{P}}$. We are free to think of $\tilde{\mathbf{P}}$ as a partial probability, in which case this describes an equilibrium: The probability distribution is unchanged by an extra step. By connectedness of the landscape, this implies that the landscape contains no terminal vacua. Conversely, suppose that the landscape has no terminal vacua. This means that, for any nontrivial initial condition $\mathbf{P}^{(0)} \neq 0$, the vector $\eta\mathbf{P}^{(0)}$ must have some nonzero components, and, in particular, their sum is nonzero. It also means that every column of η adds up to 1, so the components of $(1-\eta)\mathbf{P}$ add to zero. Thus, Eq. (2) cannot be solved.

To deal with both the terminal and the cyclic cases, I used pruned trees, which are obtained by treating the vacuum o as a terminal vacuum, except in the first step. Thus, pruned trees obey the matrix equation

$$(1 - \eta S)\mathbf{P} = \eta \mathbf{P}^{(0)}, \tag{3}$$

where S annihilates the oth column of η: $S_{ij} = \delta_{ij} - P_i^{(0)}P_j^{(0)}$. By the above proof, $(1-\eta S)$ is invertible, so Eq. (3) always has a unique solution. Thus, it is the most general matrix equation we shall require. However, in the terminal case, Eq. (2) is equivalent and more elegant.

In fact, a simpler specialized equation is also available in the cyclic case, since the pruned tree will have $P_o = P_o^{(0)}$ by conservation of the probability current. Hence, $\eta S\mathbf{P} + \eta\mathbf{P}^{(0)} = \eta(S\mathbf{P} + \mathbf{P}^{(0)}) = \eta\mathbf{P}$. Substitution into Eq. (3) yields

$$(1 - \eta)\mathbf{P} = 0. \tag{4}$$

Note that $\mathbf{P}^{(0)}$ does not appear in Eq. (4). This demonstrates that the probabilities are independent of the initial vacuum in the cyclic case, as advertised above.

To avoid confusion, let me emphasize once more that the general (pruned tree) prescription is captured by Eq. (3). It reduces to Eqs. (2) and (4) for the terminal and cyclic landscapes, respectively.

In Ref. [16], the general prescription of Garriga *et al.* [8] was applied to the special case of cyclic landscapes (for the simplest cyclic landscape, the result was first given in Ref. [14]). The probabilities obey Eq. (4), which shows that our prescription agrees with that of Garriga *et al.* if the landscape is cyclic. For terminal landscapes, such as the string landscape, the two prescriptions differ.

False vacuum eternal inflation is particularly relevant to the landscape, but it is straightforward to apply the above approach more broadly. If a worldline in slow-roll inflation (eternal or not) has a nonzero probability to end up in more than one vacuum, this can be incorporated in the matrix η. For example, the probability to end up on either side of the asymmetric double well of Ref. [7] is 50%, if the worldline starts at the top of the barrier; volume expansion factors do not enter. If a vacuum has continuous moduli, one can treat it as a (nearly) continuous set of different vacua.

Weighting.—Having defined prior probabilities, let us now ask with what probability w_i observers will emerge in vacuum i. The total probability for i to be observed is $p_i w_i/\sum_j p_j w_j$.

Anthropic arguments make sense only in a large and varied Universe, where they select for location (rather than

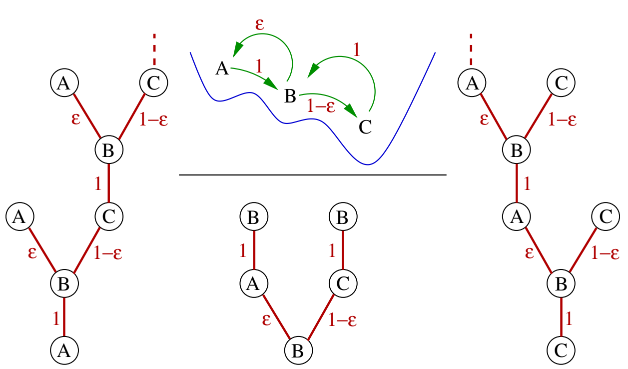

FIG. 3 (color online). A landscape without terminal vacua. For each initial vacuum, a pruned tree is shown. For example, summation over paths in the left tree yields $\mathcal{P}_A = 1$, $\mathcal{P}_B = 1/\epsilon$, and $\mathcal{P}_C = (1-\epsilon)/\epsilon$. After normalization, all pruned trees yield the same probabilities.

for initial conditions or, worse, for parameters of a fundamental theory). With a much larger cosmological constant but all other physics fixed, for example, it is plausible that life would not have formed in our part of the Universe [1].

The problem is that other parameters are far from fixed in any realistic landscape. This poses a hard optimization problem, requiring variations of the possible inflaton potentials [17,18], particle and force content [19], coupling constants, and other parameters. Moreover, the challenge of identifying conditions for "life" will be magnified, if the landscape contains low-energy theories so different from our own that we have little intuition for their impact on cosmology or condensed matter physics.

But whatever observers may consist of, they must obey the laws of causality, thermodynamics, and information theory. Observers compute; they store and retrieve information. Because the causal diamond is finite, the holographic approach makes it possible to quantify this connection: The more free energy, the more likely it is that observers will emerge. More precisely, the number of possible operations should be related to the free energy divided by the temperature at which it is burned up. This quantity is simply the increase in entropy. Thus, I propose to weight vacua by their entropy *difference* $w_i = \Delta S(i)$, defined as the entropy leaving through the top cone of the diamond minus the entropy entering the bottom cone of the diamond.

It is important to stress what this weight does *not* depend on. From a global viewpoint, it may seem natural that inflationary volume factors (which are well-defined in non-eternal slow roll) should enter directly into either the p_i or the w_i [8]. This leads to a paradox [18]: The density perturbations should be at an extreme end of the anthropically allowed window. But from a holographic point of view, volume produced in excess of one causally connected region does not boost the likelihood of a vacuum further. Inflation is useful in that it delays curvature domination, allowing more free energy to be harvested; to this extent, it will enter $\Delta S(i)$. But there is no benefit in delaying it longer than $|\Lambda|^{-1/2}$, the time when the cosmological constant begins to dominate.

Similarly, one may be tempted to include the lifetime of a metastable vacuum in its weight. But stability matters only up to a point. If the decay disrupts the harvesting of free energy, it will enter the weight factor $w_i = \Delta S$. However, lifetimes can be exponentially longer than the thermalization time scale; this does nothing to boost the probability of observers.

The entropy production in our vacuum can be estimated, and its dependence on various parameters yields constraints analogous to anthropic bounds. Unlike the latter, however, the weight $\Delta S(i)$ can plausibly be computed also for distant regions of the landscape [20], at least when averaged over many vacua. The entropy increase cannot be larger than the final entropy, which is bounded in terms of the maximal area on the future boundary of the causal diamond [10]. For de Sitter vacua, this bound is $3\pi/\Lambda$. In this sense, a small cosmological constant is better than a large one, even when other parameters scan. (Harnik, Kribs, and Perez have independently arrived at a similar conclusion.) This preference is only power law, not exponential as a purely statistical argument would imply.

Our vacuum has a positive cosmological constant, so its weight is bounded. Suppose that the landscape were infinite, in the sense that parameters could scan arbitrarily dense discretua. Then why do not we find ourselves in a region that allows for even greater complexity than our own? The landscape must be finite, and numbers such as $10^{-123} = e^{-283.2}$ [21] may turn out to be data points that will help us determine its size empirically.

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[1] S. Weinberg, Phys. Rev. Lett. **59**, 2607 (1987).
[2] R. Bousso and J. Polchinski, J. High Energy Phys. 06 (2000) 006.
[3] S. Kachru, R. Kallosh, A. Linde, and S. P. Trivedi, Phys. Rev. D **68**, 046005 (2003).
[4] L. Susskind, hep-th/0302219.
[5] F. Denef and M. R. Douglas, J. High Energy Phys. 05 (2004) 072.
[6] A. Linde, D. Linde, and A. Mezhlumian, Phys. Rev. D **49**, 1783 (1994).
[7] J. Garriga and A. Vilenkin, Phys. Rev. D **64**, 023507 (2001).
[8] J. Garriga, D. Schwartz-Perlov, A. Vilenkin, and S. Winitzki, J. Cosmol. Astropart. Phys. 01 (2006) 017.
[9] R. Easther, E. A. Lim, and M. R. Martin, J. Cosmol. Astropart. Phys. 03 (2006) 016.
[10] R. Bousso, J. High Energy Phys. 11 (2000) 038.
[11] L. Susskind, L. Thorlacius, and J. Uglum, Phys. Rev. D **48**, 3743 (1993).
[12] R. Bousso, B. Freivogel, and I.-S. Yang (to be published).
[13] L. Dyson, M. Kleban, and L. Susskind, J. High Energy Phys. 10 (2002) 011.
[14] R. Bousso, B. Freivogel, and M. Lippert, Phys. Rev. D **74**, 046008 (2006).
[15] A. Vilenkin, Phys. Rev. D **58**, 067301 (1998).
[16] V. Vanchurin and A. Vilenkin, Phys. Rev. D **74**, 043520 (2006).
[17] M. Tegmark and M. J. Rees, Astrophys. J. **499**, 526 (1998).
[18] B. Feldstein, L. J. Hall, and T. Watari, Phys. Rev. D **72**, 123506 (2005).
[19] R. Harnik, G. D. Kribs, and G. Perez, Phys. Rev. D **74**, 035006 (2006).
[20] R. Bousso, R. Harnik, G. Kribs, and G. Perez (to be published).
[21] J. Polchinski, hep-th/0603249.