Estimating Topology of Networks

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We suggest a method for estimating the topology of a network based on the dynamical evolution supported on the network. Our method is robust and can be also applied when disturbances and/or modeling errors are presented. Several examples with networks of phase oscillators, pulse-coupled Hindmarch-Rose neurons, and Lorenz oscillators are provided to illustrate our approach.

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Introduction.—The study of complex systems pervades through almost all the sciences, from cell biology to ecology, from computer science to meteorology, to name just a few. A paradigm of a complex system is a network [1] where complexity may come from different sources: topological structure, network evolution, connection and node diversity, and/or dynamical evolution. The macroscopic behavior of a network is determined by both the dynamical rules governing the nodes and the flow occurring along the links. Real networks of interacting dynamical systems—be they neurons, power stations, or lasers—are complex. The research on complex networks has been focused on the their topological structure as well as on how the topology properties of the network, such as clustering coefficient, connectivity distribution, and average network distance, influence its dynamic behavior [2–7]. For example, the effects of these properties on synchronization are well studied in the literature [8–12]. Most networks offer support for various dynamical processes. In this Letter we propose a method for determining the topological structure of a network based on the dynamical evolution supported on the network. The method is robust, can be applied to estimate the connection topology of any subnetwork, and can also be used for "online monitoring" of the dynamic evolution of the network topology.

Consider a network, which is represented by a graph. Recall that a *graph* is an ordered pair of disjoint sets (V, E) such that E is a subset of the set of unordered pairs of V. The set V is the set of *vertices* and E is the set of *edges*. The dynamical evolution on the network is given by:

$$\dot{\boldsymbol{x}}_i = \boldsymbol{f}_i(\boldsymbol{x}_i) + C\sum_{j=1}^{n} a_{ij}h_j(x_j), \tag{1}$$

where $i = 1, 2, \ldots, n$, $\boldsymbol{x}_i = [x_i, y_i, z_i, \ldots]^T \in \mathbb{R}^N$ is the state vector of node i, and $\boldsymbol{f}: \mathbb{R}^N \to \mathbb{R}^N$ describes the node equations. For simplicity only, here we assume that the first components of each node are connected to each other (more general case will be treated in another paper). Thus, $h_j(x_j)$: $\mathbb{R} \to \mathbb{R}$ is the output of the node j, and $C = [1, 0, \ldots, 0]^T$. The topology of the network connections is determined by the adjacency matrix $A = (a_{ij})$: $a_{ij} = 1$ if the node j is connected to the node i, and $a_{ij} = 0$ otherwise. The Eq. (1) can describe a network of phase oscillators [13], a network of neurons [14], or a network of chaotic oscillators. As an example of a network of phase oscillators, we consider a system of n phase oscillators,

$$\dot{\phi}_i = \omega_i + \frac{\kappa}{n}\sum_{j=1}^{n} a_{ij}\sin(\phi_j - \phi_i). \tag{2}$$

We assume that ω_i are normally distributed with mean 0 and variance 1. As an example of a network of neurons, we study a network of pulse-coupled Hindmarch-Rose (HR) neurons, for which the equation of motion is given by:

$$\begin{aligned} \dot{x}_i &= f_i(x_i, y_i, z_i) + k\sum_{j=1}^{n} a_{ij}(x_j - x_i) + g_s(x_i - V_s) \\ &\quad \times \sum_{j=1}^{n} \alpha_{ij}\Gamma(x_j), \\ \dot{y}_i &= dx_i^2 - y_i, \qquad \dot{z}_i = \mu(bx_i + c - z_i) \end{aligned} \tag{3}$$

where $f_i(\boldsymbol{x}) = ax_i^2 - x_i^3 - y_i - z_i$. The matrix (a_{ij}) is the adjacency matrix describing the electrical coupling, while the adjacency matrix (α_{ij}) describes the synaptic coupling. Finally, as an example of a network of chaotic oscillators, we consider the following array of n nonidentical Lorenz oscillators:

$$\begin{aligned} \dot{x}_i &= \sigma_i(y_i - x_i) + c\sum_{j=1}^{n} a_{ij}(x_j - x_i), \\ \dot{y}_i &= \rho x_i - x_i z_i - y_i, \qquad \dot{z}_i = x_i y_i - bz_i. \end{aligned} \tag{4}$$

Theory.—For simplicity, we consider a network of 1D oscillators, so the dynamical evolution on the network is given by:

$$\dot{x}_i = f_i(x_i) + \sum_{j\in V} a_{ij}h_j(x_j), \tag{5}$$

where $i \in V := \{1, 2, \ldots, n\}$, $x_i \in \mathbb{R}$ is the state vector of node i, and f_i: $\mathbb{R} \to \mathbb{R}$ describes the node equations. We assume that the mappings f_i and h_i are Lipchitzian for all i, that is, there exist positive constants L_{1i} and L_{2i} such that $\|f_i(y_i) - f_i(x_i)\| \leq L_{1i}\|y_i - x_i\|$ and $\|h_i(y_i) - h_i(x_i)\| \leq L_{2i}\|y_i - x_i\|$, for all i.

Assume that the functions f_i and h_i are known and x_i, for all i, can be experimentally measured (are observable). Assume further that the topology of the network is unknown. In this Letter we address a method for finding the topology of the network connections, more precisely for estimating the elements of the matrix $A = (a_{ij})$. In particular, we show that under some mild mathematical conditions, one can design control signals u_i, such that the system:

$$\begin{aligned} \dot{y}_i &= f_i(y_i) + \sum_{j=1}^{n} b_{ij}h_j(y_j) + \Delta_i(\boldsymbol{y}, b_{ij}, t) + u_i, \\ \dot{b}_{ij} &= -\gamma_{ij}h_j(y_j)(y_i - x_i), \end{aligned} \tag{6}$$

where i, $j \in V$, and γ_{ij} are positives, and can track the topology of the network. Here $\boldsymbol{y} = [y_1, \ldots, y_n]^T$ and Δ_i represent the unknown nonlinear functions (such as disturbances and modeling errors). We assume that $|\Delta_i| \leq \delta(\boldsymbol{y}, t)d(t)$, where $\delta(\boldsymbol{y}, t)$ is the known function and $d(t)$ is unknown but bounded time-varying disturbance. Let $e_i = y_i - x_i$. Consider the Lyapunov function $2\Omega = \sum_i e_i^2 + \sum_i \sum_j (1/\gamma_{ij})(b_{ij} - a_{ij})^2$. Then, we have

$$\begin{aligned} \dot{\Omega} &= \sum_{i=1}^{n} e_i\dot{e}_i + \sum_{i=1}^{n}\sum_{j=1}^{n}(b_{ij} - a_{ij})(1/\gamma_{ij})\dot{b}_{ij} \\ &= \sum_{i=1}^{n} e_i\Big\{f_i(y_i) - f_i(x_i) + \sum_j a_{ij}[h_j(y_j) - h_j(x_j)] + \Delta_i \\ &\quad + u_i + \sum_j (b_{ij} - a_{ij})h_j(y_j)\Big\} - \sum_{i=1}^{n}\sum_{j=1}^{n}(b_{ij} - a_{ij})h_j(y_j)e_i \\ &= \sum_{i=1}^{n} e_i\Big\{f_i(y_i) - f_i(x_i) \\ &\quad + \sum_j a_{ij}[h_j(y_j) - h_j(x_j)] + \Delta_i + u_i\Big\}. \end{aligned}$$

We write $L_1 = \max_i L_{1i}$ and $L_2 = \max_i L_{2i}$ and assume that u_i has the form

$$u_i = -k_1 e_i - \frac{1}{4\varepsilon_1}\delta^2 e_i.$$

Then, we have

$$\begin{aligned} \dot{\Omega} &\leq \sum_{i=1}^{n} e_i\Big[L_1 e_i + nL_2 e_i - k_1 e_i + \Delta_i - \frac{1}{4\varepsilon_1}\delta^2 e_i\Big] \\ &\leq -k\sum_i e_i^2 + \sum_i\Big(|e_i|\delta d - \frac{1}{4\varepsilon_1}\delta^2 e_i^2\Big), \end{aligned}$$

where $k = k_1 - L_1 - nL_2$ and k_1 is chosen such that $k > 0$. Noting that

$$\Big(|e_i|\delta d - \frac{1}{4\varepsilon_1}\delta^2 e_i^2\Big) = -\Big(\frac{\delta|e_i|}{2\sqrt{\varepsilon_1}} - \sqrt{\varepsilon_1}d\Big)^2 + \varepsilon_1 d^2$$

and writing $\varepsilon = n\varepsilon_1$, we finally have

$$\begin{aligned} \dot{\Omega} &\leq -k\sum e_i^2 - \sum_i\Big(\frac{\delta|e_i|}{2\sqrt{\varepsilon_1}} - \sqrt{\varepsilon_1}d\Big)^2 + \varepsilon d^2 \\ &\leq -k\sum e_i^2 + \varepsilon d^2. \end{aligned}$$

Therefore, the tracking error exponentially decays and is ultimately bounded. Since the parameters k and ε can be freely adjusted, transient performance and the final tracking accuracy are guaranteed. Therefore, $b_{ij} \approx a_{ij}$ and the system (6) tracks the topology of the network (5). We call Eq. (6) a topology estimator.

Remark 1.—The described method can be applied to any subnetwork. Indeed, let $V_1 \subset V$ be a subset of the vertex set. The outlined procedure can also be used to determine the connections within the subset V_1. Assume that for all $j \in V$, x_j can be measured. Then for all $i \in V_1$ we have

$$\begin{aligned} \dot{y}_i &= f_i(y_i) + \sum_{j\in V_1} b_{ij}h_j(y_j) + \sum_{j\in V\setminus V_1} b_{ij}h_j(y_j) + u_i, \\ &\equiv f_i(y_i) + \sum_{j\in V_1} b_{ij}h_j(y_j) + \Delta_i(\boldsymbol{y}, b_{ij}, t) + u_i, \end{aligned}$$

and one can apply the above method.

Remark 2.—In the case when the node is a high-dimensional dynamical system as for the network (1), one can follow the similar steps as for the 1D case. However, the full description of the theory in this case is beyond the scope of the Letter. The network estimator has the form:

$$\begin{aligned} \dot{\hat{\boldsymbol{x}}}_i &= f_i(\hat{\boldsymbol{x}}_i) + C\sum_{j=1}^{n} b_{ij}h_j(\hat{x}_j) + \Delta_i(\hat{\boldsymbol{x}}, b_{ij}, t) + \boldsymbol{u}_i, \\ \dot{b}_{ij} &= -\gamma_{ij}h_j(\hat{x}_j)(\hat{x}_i - x_i), \end{aligned}$$

where $i, j \in \{1, 2, \ldots, n\}$, Δ_i represent the unknown nonlinear functions, and γ_{ij} are positives. Let $\mathbf{e}_i = \hat{\boldsymbol{x}} - \boldsymbol{x}$ and $e_i = \hat{x}_i - x_i$. Let also $\boldsymbol{u}_i = \boldsymbol{u}_i^{(1)} + Cu_i^{(2)} + Cu_i^{(3)}$. Assume that one can find a control signal $\boldsymbol{u}_i^{(1)}$ such that $\mathbf{e}_i^T[f_i(\hat{\boldsymbol{x}}_i) - f_i(\boldsymbol{x}_i) + \boldsymbol{u}_i^{(1)}]$ is negative definite for all i, and h_i's are Lipchitzian, then one can follow the same steps as above [using $u_i^{(2)} = -k_1 e_i$ and $u_i^{(3)} = -1/(4\varepsilon)\delta^2 e_i$] to show that the topology of the network (1) can be also estimated with arbitrary precision.

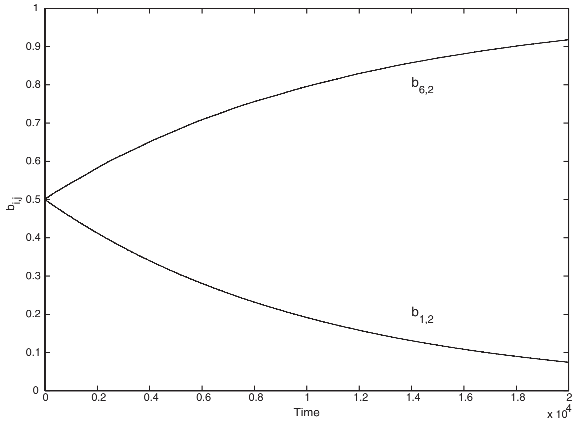

FIG. 1. Topology estimation of a network of $n = 10$ phase oscillators: $b_{1,2}$ and $b_{6,2}$ vs time.

Remark 3.—We stress that our method works also when some nodes are originally in a periodic and others in a chaotic regime. It also works when all nodes support different dynamics or when the network is in the state of partial synchronization, that is when all nodes are not synchronized; see, for example, [15].

Examples.—We now present several examples. In our first example, we consider phase oscillators (2). We investigate standard random symmetric networks, where independently for all connections $a_{ij} = 1$ with some probability and $a_{ij} = 0$ otherwise. In the numerical simulation presented here $n = 10$ and $\kappa = 0.5$. Initial conditions for the topology estimator are set to $b_{ij}(0) = 0.5$ for all i, j. Figure 1 shows the result of the numerical simulation: $b_{1,2}$ approaches correctly the value $a_{1,2} = 0$ and $b_{6,2}$ tends to the value $a_{6,2} = 1$.

In our next example we consider a network of HR oscillators (3). We assume that the neurons are identical and the synapses are fast and instantaneous. The parameter g_s is the synaptic coupling strength. The reversal potential $V_s > x_i(t)$ for all x_i and all t; i.e., the synapse is excitatory. We set $V_s = 2$. The synaptic coupling function is modeled by the sigmoidal function $\Gamma(x_j) = 1/\{1 + \exp[-10(x_j - \theta_s)]\}$, where $\theta_s = -0.25$. In the numerical example presented here: $a = 2.8$, $d = 4.4$, $c = 5$, $b = 9$, $\mu = 0.001$, $g_s = 0.34$, $k = 0.05$, and $n = 10$. Again the topology estimator correctly estimates the topology of the network, that is the matrices (a_{ij}) and (α_{ij}), as illustrated on the Fig. 2.

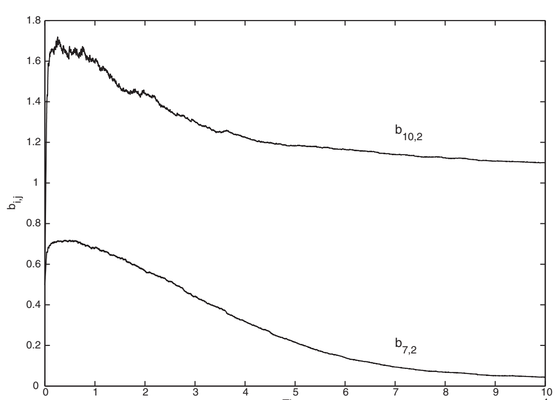

FIG. 2. Topology estimation of a network of HR oscillators: $b_{7,2}$ and $b_{10,2}$ vs time.

Finally, as a third example, we consider a network of n nonidentical Lorenz oscillators (4), for which the values of σ_i are randomly chosen in the interval [9.2; 9.4], and $\rho = 28$, $c = 0.1$, $b = 8/3$, and $n = 16$. For the numerical simulation presented below, we assume that the values of the first raw of the adjacency matrix $A = (a_{ij})$ are $a_{1,j} = 0$ for $j = 1, 6, 10, 12, 14$, and $a_{1,j} = 1$ for $j = 2, 3, 4, 5, 7, 8, 9, 11, 13, 15, 16$. Then the system

$$\dot{\hat{x}}_i = \sigma_i(\hat{y}_i - \hat{x}_i) + c\sum_{j=1}^{n} b_{ij}(\hat{x}_j - \hat{x}_i) + k(x_i - \hat{x}_i),$$

$$\dot{\hat{y}}_i = \rho\hat{x}_i - \hat{x}_i\hat{z}_i - \hat{y}_i, \qquad \dot{\hat{z}}_i = \hat{x}_i\hat{y}_i - b\hat{z}_i,$$

$$\dot{b}_{ij} = -\gamma_{ij}c(\hat{x}_j - \hat{x}_i)(\hat{x}_i - x_i),$$

where k and γ_{ij} are positives, estimates the elements of the matrix A, that is, $b_{ij} \to a_{ij}$ as time goes to infinity. Figure 3 shows the estimation of b_{ij} versus time for $i = 1$: for better visual presentation, we show $b_{ij} + j$ versus time. Note that $b_{1,5} + 5$ and $b_{1,6} + 6$ approach 6 indicating correctly that $a_{1,5} = 1$ and $a_{1,6} = 0$.

The proposed approach can be applied for online "monitoring" of the network topology. This implies that the dynamic evolution of the topological structure can be "recorded" by the online "monitor." Assume, for example, in the above network of $n = 16$ nonidentical Lorenz oscillators, we monitor only the fifth and the 11th oscillator. We assume that at $t = 400$ there is an abrupt change of the network topology: $a_{5,11} = a_{11,5} = 1$ changes to $a_{5,11} = a_{11,5} = 0$. Figure 4 shows the result of the numerical simulation of our estimator: $b_{5,11}$ and also $b_{11,5}$ estimate correctly the values of $a_{5,11}$ and $a_{11,5}$, respectively.

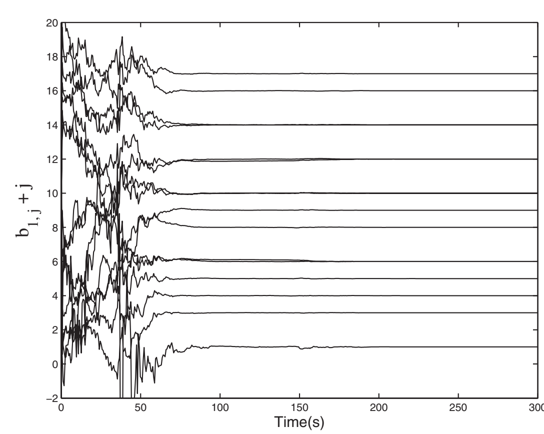

FIG. 3. Topology estimation of a network of 16 nonidentical Lorenz oscillators: $b_{1,j} + j$ vs time, for $j = 1, 2, \ldots, 16$.

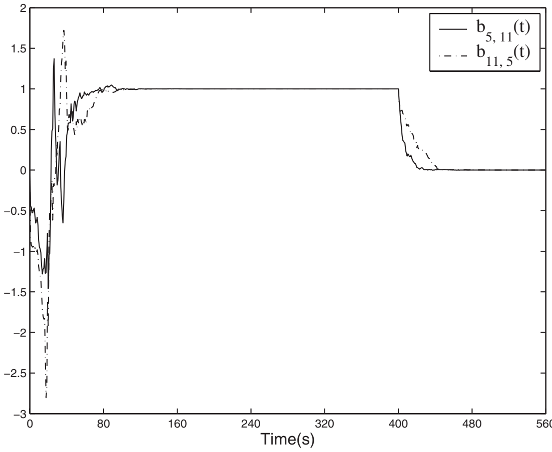

FIG. 4. Monitoring the connection between the fifth and the 11th node: $b_{5,11}$ ($b_{11,5}$) vs time.

Finally, we present an example of a topology with uncertainty. Consider a network of n nonidentical Lorenz oscillators and assume that we can only measure the variables x_i, for $i = 1, 2, \ldots, n_1$. Then the equation for our topology estimator reads:

$$\dot{\hat{x}}_i = \sigma_i(\hat{y}_i - \hat{x}_i) + c\sum_{j=1}^{n_1} b_{ij}(\hat{x}_j - \hat{x}_i) + k(x_i - \hat{x}_i) + \Delta_i,$$

$$\dot{\hat{y}}_i = \rho\hat{x}_i - \hat{x}_i\hat{z}_i - \hat{y}_i, \qquad \dot{\hat{z}}_i = \hat{x}_i\hat{y}_i - b\hat{z}_i,$$

$$\dot{b}_{ij} = -\gamma_{ij}c(\hat{x}_j - \hat{x}_i)(\hat{x}_i - x_i),$$

where $i, j \in \{1, 2, \ldots, n_1\}$, γ_{ij} are positives, and Δ_i represent the effects of the influences of the $n - n_1$ oscillators, that is $\Delta_i = \sum_{j=n_1+1}^{n} b_{ij}\hat{x}_j$. Still we can estimate the connection topology of the n_1 oscillators. Figure 5 presents the

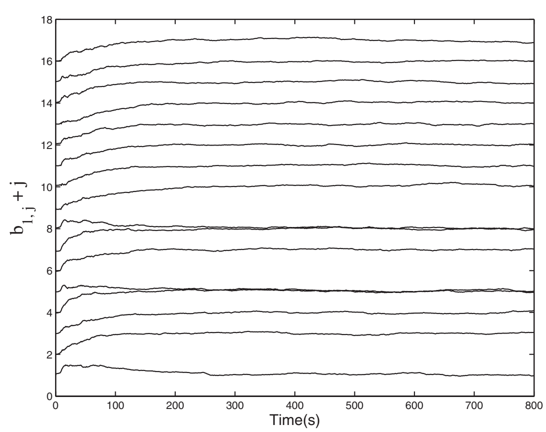

FIG. 5. Topology estimation with uncertainty: the connection topology of a subnetwork. $b_{1,j}$ vs time for the subnetwork of 16 oscillators.

results of the numerical simulation for the case $n = 17$ and $n_1 = 16$: we plot here the values of $b_{1,j} + j$ versus time for $j = 1, 2, \ldots, 16$ (assuming that $a_{1,1} = a_{1,5} = a_{1,8} = 0$ and $a_{1,j} = 1$ otherwise).

Conclusions.—In conclusion, we have suggested a method for estimating the topology of networks. We have shown that the method can be applied to estimate the connection topology in any subnetwork. In addition, we have demonstrated that the approach can be used for online monitoring the dynamic evolution of the network topology. Since our method is robust and works well with disturbances and modeling errors, we think that the method can also be applied to estimate the topology of real (sub)networks. Our approach can be easily generalized to estimate the weights of a weighted network.

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[1] A.-L. Barabási, *Linked: The New Science of Networks* (Perseus Publishing, Cambridge, Massachusetts, 2002).
[2] R. Albert and A.-L. Barabási, Rev. Mod. Phys. **74**, 47 (2002).
[3] M. E. J. Newman, SIAM Rev. **45**, 167 (2003); Phys. Rev. Lett. **89**, 208701 (2002).
[4] D. J. Watts and S. H. Strogatz, Nature (London) **393**, 440 (1998).
[5] S. H. Strogatz, Nature (London) **410**, 268 (2001).
[6] A.-L. Barabási and R. Albert, Science **286**, 509 (1999).
[7] R. Guimera *et al.*, Phys. Rev. Lett. **89**, 248701 (2002).
[8] L. M. Pecora and T. L. Carroll, Phys. Rev. Lett. **80**, 2109 (1998).
[9] M. Barahona and L. M. Pecora, Phys. Rev. Lett. **89**, 054101 (2002).
[10] T. Nishikawa *et al.*, Phys. Rev. Lett. **91**, 014101 (2003).
[11] L. Donetti, P. I. Hurtado, and M. A. Muñoz, Phys. Rev. Lett. **95**, 188701 (2005).
[12] C. Zhou, A. E. Motter, and J. Kurths, Phys. Rev. Lett. **96**, 034101 (2006).
[13] Y. Kuramoto, *Chemical Oscillations, Waves, and Turbulence* (Springer, New York, 1991); K. Wiesenfeld, P. Colet, and S. H. Strogatz, Phys. Rev. E **57**, 1563 (1998); H. Kori and A. S. Mikhailov, Phys. Rev. Lett. **93**, 254101 (2004).
[14] L. de Arcangelis, C. Perrone-Capano, and H. J. Herrmann, Phys. Rev. Lett. **96**, 028107 (2006); S. Wang, W. Wang, and F. Liu, Phys. Rev. Lett. **96**, 018103 (2006); A. J. Preyer and R. J. Butera, Phys. Rev. Lett. **95**, 138103 (2005); I. Belykh, E. de Lange, and M. Hasler, Phys. Rev. Lett. **94**, 188101 (2005); C. G. Assisi, V. K. Jirsa, and J. A. S. Kelso, Phys. Rev. Lett. **94**, 018106 (2005).
[15] C. Zhou and J. Kurths, Phys. Rev. Lett. **96**, 164102 (2006).