Two-Loop Iteration of Five-Point $\mathcal{N} = 4$ Super-Yang-Mills Amplitudes

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We confirm by explicit computation the conjectured all-orders iteration of planar maximally supersymmetric $\mathcal{N} = 4$ Yang-Mills theory in the nontrivial case of five-point two-loop amplitudes. We compute the required unitarity cuts of the integrand and evaluate the resulting integrals numerically using a Mellin-Barnes representation and the automated package of Czakon [Comput. Phys. Commun. **175**, 559 (2006)]. This confirmation of the iteration relation provides further evidence suggesting that $\mathcal{N} = 4$ gauge theory is solvable.

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In his seminal work dating to the infancy of asymptotic freedom, 't Hooft [1] gave the hope of solving quantum chromodynamics (QCD) in the so-called planar limit, when the number of colors is taken to be large. While this hope for ordinary QCD has not yet been realized, the Maldacena conjecture [2] has brought it closer for four-dimensional maximally supersymmetric Yang-Mills theory (MSYM), by proposing a duality relating it at strong coupling to type IIB string theory in five-dimensional anti–de Sitter (AdS) space at weak coupling. Heuristically, this suggests that the leading-color terms of the perturbative series should be resummable and, along with possible nonperturbative contributions, should yield relatively simple results matching those of weakly coupled gravity.

While the Maldacena conjecture does not address directly the scattering amplitudes of on-shell (massless) quanta, previous work by Anastasiou, Dixon, and two of the authors [3] shows that the basic intuition holds. That Letter presented a conjecture for an all-orders iterative structure in dimensionally regulated scattering amplitudes of MSYM. Dixon and two of the authors [4] fleshed out this structure for maximally helicity-violating (MHV) amplitudes. Witten's proposal [5] of a *weak-weak* duality between MSYM scattering amplitudes and a twistor string theory provides further indications of new structures underlying the simplicity of both MSYM and string theory in AdS space at strong world sheet coupling.

Reference [3] verified the iteration conjecture explicitly for the two-loop four-point function (a second verification was given in Ref. [6]), and Ref. [4] did so for the three-loop four-point amplitude. Furthermore, the computation of the two-loop splitting amplitude in Ref. [3], its own iteration relation, and consideration of limits as momenta become collinear shows that, were the conjecture to hold for the *five-point* two-loop amplitude, it would almost certainly hold for all MHV two-loop amplitudes. The step from four-point to five-point amplitudes is nontrivial, because at five points, functions that are not detectable in real-momentum collinear limits appear [7]. (The structure of factorization with complex momenta is not known *a priori*.)

An important step in closing this gap has recently been taken by Cachazo, Spradlin, and Volovich [8]. They confirmed the conjecture for the terms in the two-loop five-point amplitude even under parity, using an earlier guess for the integrand [9]. In this Letter, we will complete the task. We compute the integrand using the unitarity method [7,10,11], confirming the form of Ref. [9] for the parity-even terms, and providing the correct form for the parity-odd ones. We then integrate numerically at random kinematic points, using the MB integration package [12], to show that the conjecture holds for both parity-even and -odd terms. We also remark that the "unexpected iterative structure" of Ref. [8] follows from the one of Ref. [3] by setting odd parity terms to zero on both sides of the iteration formula.

The unitarity method [7,10,13] has proven powerful for computing scattering amplitudes of phenomenological and theoretical interest out of reach using conventional Feynman diagrammatic methods. Improvements [14] have followed from the use of complex momenta [5].

Perturbative amplitudes in four-dimensional massless gauge theories contain infrared singularities. These are well understood [15] in MSYM and are a subset of the ones appearing in QCD. As in perturbative QCD, the S matrix under discussion here is not the textbook one for the "true" asymptotic states of the four-dimensional theory but, rather, for states with a definite parton number. As in QCD, a summation over degenerate states would be required to obtain finite results for scattering [16]. We regulate these divergences in a supersymmetry-preserving fashion using the four-dimensional helicity (FDH) [17] variant of dimensional regularization, with $D = 4 - 2\epsilon$. (This scheme is a close relative of Siegel's dimensional reduction [18]).

We write the leading-color contributions to the L-loop $SU(N_c)$ gauge-theory n-point amplitudes as

$$\mathcal{A}_n^{(L)} = g^{n-2}\left[\frac{2e^{-\gamma\epsilon}g^2N_c}{(4\pi)^{2-\epsilon}}\right]^L \times \sum_\rho \mathrm{Tr}(T^{a_{\rho(1)}}\ldots T^{a_{\rho(n)}})A_n^{(L)}(\rho(1),\rho(2),\ldots,\rho(n)), \tag{1}$$

where γ is Euler's constant, and the sum is over noncyclic permutations of the external legs. We have suppressed the momenta and helicities k_i and λ_i, leaving only the index i as a label. This decomposition holds for all particles in the gauge supermultiplet as all are in the adjoint representation. We will find it convenient to scale out the tree amplitude, defining $M_n^{(L)}(\epsilon) \equiv A_n^{(L)}/A_n^{(0)}$.

At two loops, the iteration conjecture expresses n-point amplitudes entirely in terms of one-loop amplitudes and a set of constants [13]. For MHV amplitudes up to $\mathcal{O}(\epsilon^0)$,

$$M_n^{(2)}(\epsilon) = \tfrac{1}{2}(M_n^{(1)}(\epsilon))^2 + f^{(2)}(\epsilon)M_n^{(1)}(2\epsilon) + C^{(2)}, \tag{2}$$

where $f^{(2)}(\epsilon) = -(\zeta_2 + \zeta_3\epsilon + \zeta_4\epsilon^2 + \cdots)$, and $C^{(2)} = -\zeta_2^2/2$. Reference [4] provides analogous equations for higher-loop MHV amplitudes. Subtracting out the known infrared divergences [15] provides an all-loop form for the finite remainder, expressed in terms of the one-loop finite remainder and two constants, one of which is an anomalous dimension. A conjecture for the required anomalous dimension was presented very recently [19], based on a proposed all-loop Bethe ansatz [20]. It is rather interesting that this anomalous dimension corresponds to one of the terms appearing in the QCD one [21].

To check whether the iteration relation holds in the critical five-point case, we have evaluated a set of cuts sufficient to determine the five-gluon integrand completely. These include the three-particle cuts depicted in Fig. 1(a) as well as the contributions to the two-particle cuts from Fig. 1(b). The three-particle cuts on their own determine all integral functions, except for those which are simple products of one-loop integrals. The two-particle cuts rule out the latter (double cuts suffice).

The use of a dimensional regulator involves an analytic continuation of the loop momenta to D dimensions. At one loop, the discrepancy between treating loop momenta in four or D dimensions does not modify the amplitudes of a supersymmetric gauge theory through $\mathcal{O}(\epsilon^0)$. No such proof exists for higher loops. Thus, to ensure that no contributions are dropped, we compute the unitarity cuts in D dimensions [22]. This does complicate the analysis, because standard helicity states can no longer be used as the intermediate states. We can avoid some of the additional complexity by considering instead the $D = 10$, $\mathcal{N} = 1$ super-Yang-Mills theory. When compactified on a torus to $D = 4 - 2\epsilon$ dimensions, this is equivalent to dimensionally regulated MSYM in the FDH scheme.

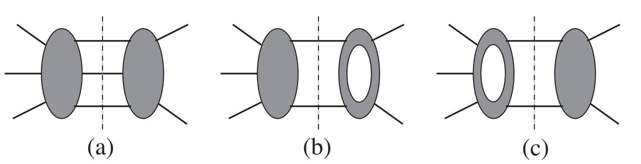

FIG. 1. The three- and two-particle cuts of the five-point amplitude.

After reducing all tensor integrals, we obtain an expression for the amplitude in terms of the integrals shown in Fig. 2. The color-ordered amplitude with four-dimensional external momenta is given by a sum over the cyclic permutations of those momenta,

$$\begin{aligned} M_5^{(2)}(\epsilon) = \frac{1}{8}\sum_{\text{cyclic}}\Bigg\{ & s_{12}^2 s_{23} I_{(a)}^{(2)}(\epsilon) + s_{12}^2 s_{15} I_{(b)}^{(2)}(\epsilon) \\ & + s_{12}s_{34}s_{45} I_{(c)}^{(2)}(\epsilon) + R\Bigg[2I_{(d)}^{(2)}(\epsilon) - 2s_{12}I_{(e)}^{(2)}(\epsilon) \\ & + \frac{s_{12}}{s_{34}s_{45}}\left(\frac{\delta_{-++}}{s_{23}}I_{(b)}^{(2)}(\epsilon) - \frac{\delta_{-+-}}{s_{51}}I_{(a)}^{(2)}(\epsilon)\right) \\ & + \frac{\delta_{+-+}}{s_{23}s_{51}}I_{(c)}^{(2)}(\epsilon)\Bigg]\Bigg\}. \end{aligned} \tag{3}$$

Here $s_{ij} = (k_i + k_j)^2$, $R = \varepsilon_{1234}s_{12}s_{23}s_{34}s_{45}s_{51}/G_{1234}$,

$$\begin{aligned} \delta_{abc} &= s_{12}s_{51} + as_{12}s_{23} + bs_{23}s_{34} - s_{51}s_{45} + cs_{34}s_{45}, \\ \varepsilon_{1234} &= 4i\varepsilon_{\mu\nu\rho\sigma}k_1^\mu k_2^\nu k_3^\rho k_4^\sigma = \mathrm{Tr}[\gamma_5 \not{k}_1 \not{k}_2 \not{k}_3 \not{k}_4], \end{aligned} \tag{4}$$

and $G_{1234} = \det(s_{ij})$ $(i, j = 1, \ldots, 4)$. (In δ, $a, b, c = \pm 1$.) The terms lacking a factor of ε_{1234} are even under parity, while those with such a factor are odd. The even terms match the guess originally given in Ref. [9], but the odd terms differ (the odd terms in Ref. [9] do match the four-dimensional double two-particle cuts).

Because of the $1/\epsilon^2$ infrared singularity in one-loop amplitudes, and because these appear squared in the iteration relation, we need expressions valid through $\mathcal{O}(\epsilon^2)$. A representation of the one-loop five-point amplitude, ex-

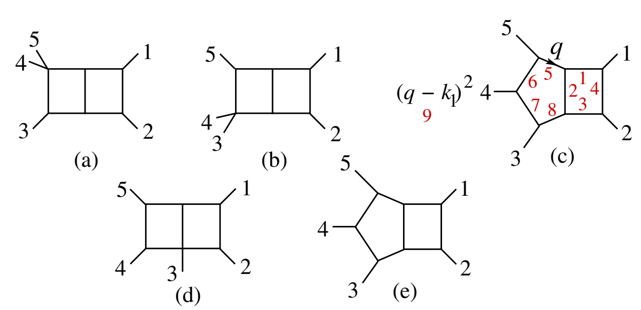

FIG. 2 (color online). The two-loop integrals appearing in the five-point amplitude, with all external momenta flowing outwards. The normalization is as given in Eq. (8), and the numerical labels on the internal propagators in (c) specify the arbitrary powers a_i. The prefactor in (c) is understood to be inserted in the numerator with power $-a_9$; in Eq. (3), $-a_9 = 1$.

tending Ref. [7] to all orders in ϵ, may be found in Ref. [23]:

$$M_5^{(1)}(\epsilon) = -\frac{1}{4}\sum_{\text{cyclic}} s_{12}s_{23}I_{(a)}^{(1)}(\epsilon) - \frac{\epsilon}{2}\varepsilon_{1234}I_{(b)}^{(1)6-2\epsilon}(\epsilon), \quad (5)$$

in terms of the integrals of Fig. 3. As indicated by the superscript, the second integral [Fig. 3(b)] is to be evaluated in $6-2\epsilon$ dimensions. In $D=6$, it is completely finite, but because it appears multiplied by an infrared-singular integral in Eq. (2) we need its value through $\mathcal{O}(\epsilon)$.

To obtain Laurent expansions in ϵ for our integrals, we use the Mellin-Barnes (MB) technique, successfully applied in numerous calculations (see, e.g., Refs. [4,24–27] and Chap. 4 of Ref. [28]). It relies on the identity

$$\frac{1}{(X+Y)^\lambda} = \int_{\beta-i\infty}^{\beta+i\infty}\frac{Y^z}{X^{\lambda+z}}\frac{\Gamma(\lambda+z)\Gamma(-z)}{\Gamma(\lambda)}\frac{dz}{2\pi i}, \quad (6)$$

where $-\mathrm{Re}\lambda < \beta < 0$. This basically replaces a sum over terms raised to some power with a product of factors.

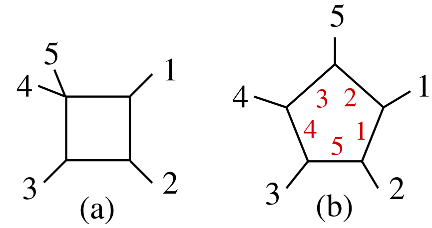

FIG. 3 (color online). The one-loop integrals required to all orders in ϵ for the one-loop five-point amplitude. The normalization is as given in Eq. (7), and the numerical labels on the internal propagators in (b) specify the arbitrary powers a_i.

The box function in Fig. 3(a) was given to all orders in ϵ in terms of a hypergeometric function in Ref. [29]. Here we need its value through $\mathcal{O}(\epsilon^2)$. Evaluating the pentagon in Fig. 3(b) with arbitrary powers of propagators also allows a parallel evaluation of this integral to the required order.

The derivation of a fourfold MB representation for the one-loop pentagon diagram is straightforward, after Feynman parametrizing,

$$\begin{aligned}
P^{(1)}(a_1,\ldots,a_5;s_{12},\ldots,s_{51};\epsilon) &= -ie^{\gamma\epsilon}(4\pi)^{D/2}\int\frac{d^D\ell}{(2\pi)^D}\frac{1}{[\ell^2]^{a_1}[(\ell+k_1)^2]^{a_2}[(\ell+K_{15})^2]^{a_3}[(\ell-K_{23})^2]^{a_4}[(\ell-k_2)^2]^{a_5}}\\
&= \frac{e^{\gamma\epsilon}(-1)^A}{\Gamma(4-A-2\epsilon)}\frac{1}{\prod_{j=1}^5\Gamma(a_j)}\int_{-i\infty}^{+i\infty}\prod_{j=1}^4\Gamma(-z_j)\frac{dz_j}{2\pi i}\frac{(-s_{45})^{z_1}(-s_{34})^{z_2}(-s_{23})^{z_3}(-s_{12})^{z_4}}{(-s_{15})^{A+\epsilon-2+z_{1234}}}\\
&\quad\times\Gamma(a_2+z_{14})\Gamma(A+\epsilon-2+z_{1234})\Gamma(2-\epsilon-a_{2345}-z_{124})\Gamma(2-\epsilon-a_{1245}-z_{134})\\
&\quad\times\Gamma(a_4+z_{13})\Gamma(a_5+z_{24}),
\end{aligned}\quad (7)$$

where $K_{ij}=k_i+k_j$, $a_{2345}=a_2+a_3+a_4+a_5$, $A=\sum a_i$, $z_{124}=z_1+z_2+z_4$, etc. We have allowed for arbitrary powers of propagators so that we can obtain all one-loop integrals. Taking $a_5\to 0$, with other $a_i=1$, gives the box integral $I_{(a)}^{(1)}$ in Fig. 3(a). Setting all $a_i=1$ and shifting all terms except the $e^{\gamma\epsilon}$ prefactor by $\epsilon\to\epsilon-1$ yields the $D=6-2\epsilon$ pentagon $I_{(b)}^{(1)6-2\epsilon}$, corresponding to Fig. 3(b). The contours of integration are chosen so that the real parts of the arguments of all gamma functions are positive.

The various two-loop pentabox integrals have a sevenfold MB representation obtained by inserting a threefold MB representation for a two-mass double box into Eq. (7):

$$\begin{aligned}
P^{(2)}(\{a_i\};\{s_{ij}\};\epsilon) = {}&\frac{e^{2\gamma\epsilon}(-1)^A}{\prod_{j=1,2,3,4,6,7}\Gamma(a_j)\Gamma(4-a_{1234}-2\epsilon)}\int_{-i\infty}^{+i\infty}\prod_{j=1}^7\Gamma(-z_j)\frac{dz_j}{2\pi i}\frac{(-s_{45})^{z_1}(-s_{12})^{2-a_{1234}-\epsilon+z_4-z_{567}}(-s_{23})^{z_3}(-s_{34})^{z_2}}{(-s_{15})^{a_{56789}+\epsilon-2+z_{1234}-z_{567}}}\\
&\times\frac{\Gamma(a_7+z_{13})\Gamma(a_5+z_{14}-z_5)\Gamma(a_8+z_{24}-z_6)}{\Gamma(a_5-z_5)\Gamma(a_8-z_6)\Gamma(a_9-z_7)}\\
&\times\frac{\Gamma(2-\epsilon-a_{5678}-z_{124}+z_{56})\Gamma(a_4+z_7)\Gamma(a_2+z_{567})\Gamma(a_{56789}+\epsilon-2+z_{1234}-z_{567})}{\Gamma(4-2\epsilon-a_{56789}+z_{567})}\\
&\times\Gamma(2-\epsilon-a_{124}-z_{57})\Gamma(2-\epsilon-a_{234}-z_{67})\Gamma(a_{1234}+\epsilon-2+z_{567})\Gamma(2-\epsilon-a_{5789}-z_{134}+z_{567}).
\end{aligned}\quad (8)$$

The limit $a_6\to 0$ or $a_7\to 0$ with $a_9=0$ and the other $a_i=1$ yields the double box with one massive leg [Figs. 2(a) and 2(b)] in agreement with Refs. [27,30]. Moreover, $P^{(2)}(1,\ldots,1,-1)$, $P^{(2)}(1,\ldots,1,0)$, and $P^{(2)}(1,\ldots 1,0,0)$ yield the integrals in Figs. 2(c), 2(e), and 2(d), respectively.

An essential step in the use of the MB technique is the resolution of singularities in ϵ or zeros that appear as $a_i\to 0$. There are two strategies for doing this [24,25]. Quite recently, the second strategy was formulated algorithmically [12,31] and implemented in the MB package [12]. It produces code that allows the integrals to be evaluated numerically to reasonably high accuracy.

The even terms in Eq. (3) were recently evaluated in Ref. [8] using the MB package [12] along with the guess of Ref. [9]. Those authors stated that an "unexpected iterative structure" holds for the parity-even terms alone. We may observe that this structure is not independent of the complete iteration formula Eq. (2): Use the results for the one-loop five-point amplitude in Eq. (5), set the odd terms to zero, and use the fact that the one-loop MHV amplitudes have even parity through $\mathcal{O}(\epsilon^0)$. At higher loops, we do not expect a clean separation between "even" and "odd" terms, as nonvanishing terms of the form ε_{1234}^2 will arise. These are even under parity.

We have evaluated all the two-loop integrals in Fig. 2 through $\mathcal{O}(\epsilon^0)$ and the one-loop integrals in Fig. 3 through $\mathcal{O}(\epsilon^2)$ using the representations in Eqs. (7) and (8). We have checked to a numerical accuracy of five significant digits at three independent and generic kinematic points that the iteration formula (2) is indeed correct for the complete amplitude. [Obtaining this numerical accuracy is straightforward, because we find no large cancellations between the terms in Eq. (3)]. This is a crucial check on the conjecture, because the parity-odd terms in the five-point amplitude are precisely the ones which are not constrained by collinear factorization onto four-point amplitudes. A fully analytic confirmation would also be desirable. Techniques such as those of Ref. [6] may be useful in this regard.

The calculation presented here makes a nontrivial addition to the existing body of evidence for the iteration conjecture [3,4]. The conjecture itself gives us good reasons to believe that MSYM is solvable. Within the context of the planar perturbative expansion, this would imply the resummability of the series. Parallel developments in uncovering the integrable structure of the theory (see, e.g., Refs. [20,32]) also lend credence to this belief.

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