Supersymmetric 3D Anti–de Sitter Space Solutions of Type IIB Supergravity

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For every positively curved Kähler-Einstein manifold in four dimensions, we construct an infinite family of supersymmetric solutions of type IIB supergravity. The solutions are warped products of AdS_3 with a compact seven-dimensional manifold and have nonvanishing five-form flux. Via the anti–de Sitter/conformal field theory correspondence, the solutions are dual to two-dimensional conformal field theories with (0, 2) supersymmetry. The corresponding central charges are rational numbers.

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The anti–de Sitter/conformal field theory (AdS/CFT) correspondence [1] states that any solution of string or M theory with an AdS_{d+1} factor should be equivalent to a conformal field theory in d spacetime dimensions. This correspondence, and its generalizations, has provided profound insight into the nonperturbative structure of string theory, the structure of quantum field theory, and the quantum properties of black holes.

Backgrounds with AdS_3 factors are of particular interest because, unlike in higher dimensions, the conformal group in two dimensions is infinite dimensional. As a consequence, two-dimensional conformal field theories are much more tractable than their higher dimensional cousins; for instance, many models are exactly solvable, and there is a considerable literature on the subject. It would be a significant development if, via the AdS/CFT correspondence, string or M theory can make contact with this large body of work.

However, until now there were only a few known explicit $AdS_3 \times \mathcal{M}$ solutions, with compact $\mathcal{M}$. The most well studied class of examples are the $AdS_3 \times S^3 \times X$ backgrounds of type IIB supergravity, where $X = T^4$ or $K3$. These are dual to $\mathcal{N} = (4, 4)$ conformal field theories that are deformations of the sigma model based on the orbifold $Sym(X)^n/S_n$. From a string theory perspective, these backgrounds describe the backreaction of a D-brane configuration that can be related to a black hole in five dimensions. It is a remarkable fact that the entropy of this black hole can be precisely derived from the central charge of the dual conformal field theory [2].

There have also been recent investigations into the conformal field theory dual to the $AdS_3 \times S^3 \times S^3 \times S^1$ background of type II string theory [3] (see [4–8] for earlier discussions). Despite the fact that the field theory has a larger version of $\mathcal{N} = (4, 4)$ superconformal symmetry than those dual to the $AdS_3 \times S^3 \times X$ solutions, it has proved more difficult to identify them as a number of subtleties arise.

The purpose of this Letter is to present a new infinite class of supersymmetric AdS_3 backgrounds of type IIB string theory, which are dual to two-dimensional conformal field theories with $\mathcal{N} = (0, 2)$ supersymmetry. It will be very interesting if these conformal field theories can be explicitly identified. It will also be very interesting to know whether our solutions can be related to black holes.

The new solutions are warped products of AdS_3 with a compact seven-dimensional manifold $\mathcal{M}_7$ and have nontrivial self-dual five form. The manifold $\mathcal{M}_7$ is constructed as a $U(1)$ fibration over a six-dimensional manifold B_6. In turn, B_6 is an S^2 bundle over an arbitrary Kähler-Einstein manifold KE_4 with positive curvature. Such KE_4 manifolds are either $S^2 \times S^2$, $\mathbb{CP}^2$, or a del Pezzo surface dP_k with $k = 3, \ldots, 8$. For each such KE_4 we have an infinite discrete number of explicit solutions parametrized by two positive integers p and q, together with an integer n which specifies the D3-brane charge. The fibration structure implies that the group of symmetries preserving the solutions contains at least two $U(1)$ factors, one of which corresponds to the R symmetry of the dual conformal field theory. The construction is remarkably similar to the construction of seven-dimensional Sasaki-Einstein manifolds presented in [9], but we do not know of any direct connection.

As we shall show, the standard supergravity computation gives a rational central charge c for the dual two-dimensional superconformal field theories. Specifically,

$$c = \frac{9pq^2(p + mq)}{3p^2 + 3mpq + m^2q^2} \frac{Mq}{m^2h^2} n^2, \qquad (1)$$

where the integers m and M depend on the specific choice of KE_4: for $S^2 \times S^2$ we have $m = 2$, $M = 8$; for $\mathbb{CP}^2$ we have $m = 3$, $M = 9$; for the del Pezzos dP_k, we have $m = 1$, $M = 9 - k$. Finally, $h = \text{hcf}\{M/m^2, q\}$.

The type IIB solutions presented here were constructed from a much richer set of solutions of $D = 11$ supergravity that will be described elsewhere [10]. The latter solutions are warped products of AdS_3 with eight-dimensional manifolds that are topologically S^2 bundles over six-dimensional Kähler spaces. In the special case that the six-dimensional manifold is $KE_4 \times T^2$, dimensional reduction along one leg of the T^2 and T duality along the other leg leads to the solutions presented here. In the companion paper [10] we will also show that when the six-dimensional manifold is $S^2 \times S^2 \times T^2$, with the S^2 having different radii, we obtain additional generalizations of the type IIB solutions presented here. The construction of these new AdS_3 solutions has many similarities with the construction of the AdS_5 solutions constructed in [11,12]. Indeed the latter references provided key inspiration for the work presented here and in [10].

In the remainder of this Letter we present the detailed local form of the new solutions and then determine the conditions that need to be imposed in order for the local solutions to extend to global solutions.

The local solutions.—The type IIB solutions have a metric that is a warped product of AdS_3 with a seven-dimensional manifold $\mathcal{M}_7$:

$$ds^2 = L^2 w[ds^2(\mathrm{AdS}_3) + ds^2(\mathcal{M}_7)]. \qquad (2)$$

The warp factor, w, just depends on the coordinates on $\mathcal{M}_7$, and hence this metric has all of the isometries of $ds^2(\mathrm{AdS}_3)$. If KE_4 is an arbitrary positively curved Kähler-Einstein manifold with metric $ds^2_{\mathrm{KE}_4}$ and Kähler form J, then the metric on $\mathcal{M}_7$ is given by

$$ds^2(\mathcal{M}_7) = \frac{3}{8y} ds^2_{\mathrm{KE}_4} + \frac{9dy^2}{4q(y)} + \frac{q(y)D\psi^2}{16y^2(y^2 - 2y + a)} + \frac{y^2 - 2y + a}{4y^2} Dz^2, \qquad (3)$$

where $D\psi = d\psi + P$, $Dz = dz - g(y)D\psi$, and

$$g(y) = \frac{a - y}{2(y^2 - 2y + a)}, \quad q(y) = 4y^3 - 9y^2 + 6ay - a^2. \qquad (4)$$

Here a is a constant, $\mathrm{dP} = J$, and in these coordinates the warp factor is simply $w = y$. We have chosen normalizations so that $ds^2_{\mathrm{AdS}_3}$ is the metric on a unit radius AdS_3 and $\mathcal{R} = J$, where $\mathcal{R}$ is the Ricci form of KE_4 and the constant L is arbitrary, reflecting the scaling symmetry of the type IIB supergravity action. The only other nonzero type IIB field in the solution, other than the string coupling g_s, is the self-dual five form which can be written as

$$g_s F_5 = L^4[\mathrm{vol}_{\mathrm{AdS}_3} \wedge \omega_2 + J \wedge \omega_3], \qquad (5)$$

where

$$\omega_2 = -\frac{a}{4}J + \frac{y(a - y)}{2(y^2 - 2y + a)} dy \wedge D\psi + y dy \wedge Dz,$$
$$\omega_3 = \frac{3(y - a)}{64y} J \wedge Dz + \frac{3a}{64y^2} dy \wedge D\psi \wedge Dz + \frac{3q(y)}{128y(y^2 - 2y + a)} J \wedge D\psi, \qquad (6)$$

and $\mathrm{vol}_{\mathrm{AdS}_3}$ is the volume form of $ds^2(\mathrm{AdS}_3)$. Note that both ∂_ψ and ∂_z are Killing vectors, and thus the symmetry group of the background, including F_5, is at least $G \times U(1)^2$, where G is the group of the isometries of KE_4 that preserve J.

In [10] we show how to derive this class of solutions from a more general family of solutions of $D = 11$ supergravity. We also explicitly discuss the preservation of supersymmetry arguing that the solutions must be dual to conformal field theories with $\mathcal{N} = (0, 2)$ supersymmetry. Furthermore, the form of the Killing spinors implies that ∂_ψ generates the isometry dual to the $U(1)_R$ symmetry of the field theory. Here, instead, we show that we do indeed have a solution by simply comparing with the elegant analysis of the most general type IIB supergravity solutions with AdS_3 factors and nonvanishing F_5 presented in [13]. There it was shown that the metric $ds^2(\mathcal{M}_7)$ can always, locally, be written as a $U(1)$ fibration over a six-dimensional Kähler manifold satisfying some additional properties. Introducing the new coordinates $\psi = \psi' - 2z'$, $z = -2z'$ and identifying z' as the coordinate on the $U(1)$ fibration, one can check that our solution satisfies all of the conditions in [13] [one needs to take into account a rescaling of the five-form flux and also a typo in (3.22) of [13]].

Global analysis.—We now need to show that the local solution given above can be defined globally. First, we need to fix the global structure of $\mathcal{M}_7$. We will assume that $\mathcal{M}_7$ is an S^1 bundle (with the fiber parametrized by z) over a compact six-dimensional base manifold, B_6. The metric on B_6 is given by

$$ds^2(B_6) = \frac{3}{8y} ds^2_{\mathrm{KE}_4} + \frac{9dy^2}{4q(y)} + \frac{q(y)D\psi^2}{16y^2(y^2 - 2y + a)}. \qquad (7)$$

For a suitable choice of the range of a and y, one can take B_6 to be an S^2 bundle (with the fiber parametrized by y, ψ) over KE_4. More precisely, if $\mathcal{L}$ is the canonical line bundle of KE_4, the S^2 bundle is obtained by adding a point to each fiber. Topologically, $\mathcal{M}_7$ is the same manifold that was used in the construction of seven-dimensional Sasaki-Einstein metrics found in [9].

We first need to show that the metric (7) on B_6 is complete and regular. It has potentially singular points at the roots of the cubic polynomial $q(y)$, at the roots of the quadratic polynomial $y^2 - 2y + a$, and at $y = 0$. If we assume that $a \in (0, 1)$, then the three roots y_i of $q(y)$ are real and strictly positive. If we let $y_1 < y_2 < y_3$, then $y_1, y_2 \in (0, 1)$. Furthermore, $y^2 - 2y + a$ is strictly positive in the interval (y_1, y_2). Thus, by choosing the range of $y \in (y_1, y_2)$ we are left with potential problems only at y_1, y_2, where g_{yy} diverges and $g_{\psi\psi}$ vanishes. However, these are merely coordinate singularities analogous to those of polar coordinates at the origin of $\mathbb{R}^2$. Near y_1 and y_2 (and, in fact, also y_3) the (y, ψ) part of the metric takes the approximate form

$$\frac{9}{4q'(y_i)}\left[dr^2 + \frac{q'(y_i)^2}{144y_i^2(y_i^2 - 2y_i + a)}r^2 d\psi^2\right], \tag{8}$$

where we defined $r_i = 2\sqrt{y - y_i}$. The observation that $q'(y)^2 - 144y^2(y^2 - 2y + a) = -36q(y)$ for any y shows that (8) is free from conical singularities if the period of ψ is chosen to be $\Delta\psi = 2\pi$. Thus the local metric is regular everywhere in B_6 if we restrict $a \in (0, 1)$ and $y \in (y_1, y_2)$. These choices also ensure that the warp factor $w = y$ does not vanish or diverge, which would have led to singularities in the full ten-dimensional solution.

We now turn to showing that the full metric (3) is consistent with $\mathcal{M}_7$ being a $U(1)$ bundle over B_6. Observe first that the norm of the Killing vector ∂_z never vanishes (or diverges) and so the size of the S^1 fiber is always finite. Let us write $Dz = dz - A$ and denote the period of z by $\Delta z = 2\pi l$. For the metric to be well defined the rescaled one form $l^{-1}A$ must be a connection on a *bona fide* $U(1)$ fibration. This is equivalent to the condition that the corresponding first Chern class $\frac{1}{2\pi}l^{-1}dA$ lies in the integer cohomology $H^2_{\text{de Rham}}(B_6, \mathbb{Z})$. [Generically, the first Chern class may include torsion elements in $H^2(B_6, \mathbb{Z})$, but as discussed in [9], here $\pi_1(B_6) = 0$ so there is no torsion and $H^2(B_6, \mathbb{Z}) \cong H^2_{\text{de Rham}}(B_6, \mathbb{Z})$.] We observe first that

$$dA = g(y)J + g'(y)dy \wedge D\psi \tag{9}$$

is a globally defined two form on B_6: the first term is a smooth polynomial times the globally defined Kähler form, and the second is a smooth polynomial times $dy \wedge D\psi$. The latter two form could only be singular at the roots y_1, y_2, but near those points it is approximately $rdr \wedge d\psi$, which is the volume form on $\mathbb{R}^2$ in polar coordinates.

The condition that the first Chern class is in $H^2_{\text{de Rham}}(B_6, \mathbb{Z})$ is equivalent to requiring that the corresponding periods are integral, that is,

$$P(C) = \frac{1}{2\pi}\int_C l^{-1}dA_1 \in \mathbb{Z}, \tag{10}$$

for any curve $C \in H_2(B_6, \mathbb{Z})$. To check this we need a basis of the free part of $H_2(B_6, \mathbb{Z})$. In fact, such a basis is described in [9] in a very similar setting. Let $\{\Sigma_a\}$ be a basis for the free part of $H_2(\text{KE}_4, \mathbb{Z})$. Then the $\{C_0, C_a\}$ form a basis of the free part of $H_2(B_6, \mathbb{Z})$, where we take C_0 to be the fiber S^2 at a fixed point in the KE_4 base space, and $\{C_a\}$ to be the two-cycles $\{\Sigma_a\}$ sitting at one of the poles of the S^2, say, $y = y_1$. We find that

$$P(C_0) = l^{-1}[g(y_2) - g(y_1)], \qquad P(C_a) = l^{-1}g(y_1)mn_a, \tag{11}$$

where m and n_a are integers related to $\mathcal{L}$, the canonical line bundle of KE_4. (One might also consider the periods over two-cycles at the other pole $\{\tilde{C}_a\}$; however, these are not independent, since the S^2 fibration is such that as homology classes $\tilde{C}_a = C_a + mn_aC_0$.) Specifically, m is the largest positive integer m (known as the Fano index) for which there is a line bundle $\mathcal{N}$ such that $\mathcal{L} = \mathcal{N}^m$, and n_a are the periods $n_a = \int_{\Sigma_a} c_1(\mathcal{N})$ of the Chern class of $\mathcal{N}$. By construction the n_a are coprime. [In what follows it is useful to also write the homology class $\Sigma_{\mathcal{N}}$, the Poincaré dual of $c_1(\mathcal{N})$, as $\Sigma_{\mathcal{N}} = s^a\Sigma_a$. Again by definition the s^a are coprime.] It is then easy to see that the periods $P(C)$ are integer if and only if

$$g(y_2) - g(y_1) = lq, \qquad g(y_1) = lp/m, \tag{12}$$

for some integers $p, q \in \mathbb{Z}$. If p and q are relatively prime, the periods have no common factors and $\mathcal{M}_7$ is simply connected. Note that in general $g(y_2)/g(y_1) = (p + mq)/p$ is rational.

To analyze these conditions it is convenient to introduce a different variable, $x = (4y - a)/3a$, in terms of which the cubic polynomial reads

$$q(y(x)) = \frac{a^2}{16}[a(1 + 3x)^3 - (1 - 9x)^2]. \tag{13}$$

Some algebra then leads to

$$\sqrt{x(y_1)} = [1 + 2\tfrac{g(y_2)}{g(y_1)}]^{-1}. \tag{14}$$

It can be checked that for any $a \in (0, 1)$, the location of the first root is such that $x(y_1) < 1/9$, and so the condition (14) implies that we can take $p, q > 0$. Using (13) we find that the parameter a must be rational and of the form

$$a = \frac{m^2q^2(3p + mq)^2(3p + 2mq)^2}{4(3p^2 + 3mpq + m^2q^2)^3} \tag{15}$$

and

$$l = \frac{2(3p^2 + 3mpq + m^2q^2)m}{3p(p + mq)(2p + mq)}. \tag{16}$$

To summarize, we have shown that for each pair of integers p, q with $p, q > 0$, the background (2), (3), and (5) gives a regular supersymmetric type IIB supergravity solution with compact $\mathcal{M}_7$ provided that the parameter a and the parameter fixing the period of the z circle are given by (15) and (16), respectively. If p and q are coprime, $\mathcal{M}_7$ is simply connected.

Flux quantization and central charge.—To ensure that we have a good solution of string theory we need to show that the five-form flux is globally defined on $\mathcal{M}_7$ and furthermore is quantized. The appropriate quantization condition is that the periods of F_5 are integers:

$$N(D) = \frac{1}{(2\pi l_s)^4}\int_D F_5 \in \mathbb{Z}, \tag{17}$$

for any five-cycle $D \in H_5(\mathcal{M}_7, \mathbb{Z})$, where l_s is the string length.

Recall that we already argued that $dy \wedge D\psi$ is globally defined, as is dy since it is proportional to rdr at the roots y_1 and y_2, while by definition J and Dz are global forms. Given the expression (6), we immediately see that F_5 is globally defined. To check that the periods are quantized we need a basis for the free part of $H_5(\mathcal{M}, \mathbb{Z})$. Note first that a basis for the free part of $H_4(B_6, \mathbb{Z})$ is given by a section of the S^2 bundle over KE_4, say, at y_1 or y_2 together with the S^2 fibrations over each basis two-cycle $\Sigma_a \in H_2(\mathrm{KE}_4, \mathbb{Z})$. Since the $U(1)$ bundle over B_6 is nontrivial, all nontrivial five-cycles come from the $U(1)$ fibration over a four-cycle in B_6. Let us label these as follows: D_0 denotes the five-cycle arising from the section $y = y_1$, $\tilde{D}_0$ is the cycle corresponding to $y = y_2$, and D_a the cycle arising from Σ_a. Note that these cycles are not independent. From the S^2 fibration structure of B_6 we have $D_0 = \tilde{D}_0 + ms^a D_a$, while, using similar arguments to those in the appendix of [12], the $U(1)$ fibration is such that $q\tilde{D}_0 + ps^a D_a = 0$. The periods of F_5 are given by

$$\begin{aligned} N(D_0) &= -\left(\frac{3L^4}{64\pi l_s^4 g_s}\right)\left(\frac{m}{p(p+mq)}\right)M(p+mq), \\ N(\tilde{D}_0) &= -\left(\frac{3L^4}{64\pi l_s^4 g_s}\right)\left(\frac{m}{p(p+mq)}\right)Mp, \\ N(D_a) &= \left(\frac{3L^4}{64\pi l_s^4 g_s}\right)\left(\frac{m^3}{p(p+mq)}\right)qn_a, \end{aligned} \tag{18}$$

where M is a positive integer depending on the choice of KE_4, given by

$$M = \int_{\mathrm{KE}_4} c_1(\mathcal{L}) \wedge c_1(\mathcal{L}) = \frac{1}{4\pi^2}\int_{\mathrm{KE}_4} \mathcal{R} \wedge \mathcal{R} = m^2 s^a n_a.$$

These expressions reflect the relations between the five-cycles mentioned above. The flux quantization condition, which is a quantization for the possible AdS_3 radii in string units, is thus

$$\frac{3L^4}{64\pi l_s^4 g_s} = \frac{p(p+mq)}{hm^3} n, \tag{19}$$

where n is an arbitrary integer, we are assuming that p and q are coprime and recall that $h = \mathrm{hcf}\{M/m^2, q\}$.

Since the solutions have only nonvanishing five-form flux, it is natural to interpret them as the near horizon limit of some configuration of wrapped and/or intersecting D3-branes after taking into account their backreaction. The minimal value of $n = 1$ would then naturally correspond to the minimal configuration of D3-branes, with higher values of n corresponding to the backreacted geometries of n coincident configurations of such D3-branes. From (19) we see that, as is standard in the AdS/CFT correspondence, for finite p and q, one can have small g_s together with small curvatures only if $n \gg 1$.

Having established all the conditions for our solutions to be proper string theory backgrounds, we now calculate the central charge of the dual conformal field theories. It is well known [14] that the central charge c is fixed by the AdS_3 radius L and the Newton constant $G_{(3)}$ of the effective three-dimensional theory obtained by compactifying type IIB supergravity on $\mathcal{M}_7$:

$$c = \frac{3L}{2G_{(3)}}. \tag{20}$$

In our conventions, the type IIB supergravity Lagrangian reads

$$\frac{1}{(2\pi)^7 g_s^2 l_s^8}\sqrt{-\det g}R + \cdots. \tag{21}$$

Integrating this term over $\mathcal{M}_7$ gives the effective $G_{(3)}$ and hence the rational central charges given in (1). Note that the n dependence of (1) is consistent with the comment that the solution describes n copies of a minimal D3-brane configuration, the n^2 degrees of freedom arising from open strings ending on the n branes.

The solutions with $\mathrm{KE}_4^+ = CP^2$ or $S^2 \times S^2$ have global symmetries that include a $U(1) \times U(1)$ factor that leaves the Killing spinors invariant. As a consequence the dual CFTs will have exactly marginal β deformations for which corresponding supergravity solutions can be constructed using the technique of [15]. The CFTs dual to the solutions with $\mathrm{KE}_4^+ = \mathrm{dP}_k$ with $k > 4$ have exactly marginal deformations corresponding to the deformations of the complex structure of dP_k. Thus the only potentially isolated CFTs are those dual to the solutions with $\mathrm{KE}_4^+ = \mathrm{dP}_3$ and dP_4.

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