Initial Value Problem Solution of Nonlinear Shallow Water-Wave Equations

Utku Kânoğlu
Department of Engineering Sciences, Middle East Technical University, 06531 Ankara, Turkey

Costas Synolakis
School of Engineering, University of Southern California, Los Angeles, California 90089-2531, USA
(Received 28 February 2006; published 4 October 2006)

The initial value problem solution of the nonlinear shallow water-wave equations is developed under initial waveforms with and without velocity. We present a solution method based on a hodograph-type transformation to reduce the nonlinear shallow water-wave equations into a second-order linear partial differential equation and we solve its initial value problem. The proposed solution method overcomes earlier limitation of small waveheights when the initial velocity is nonzero, and the definition of the initial conditions in the physical and transform spaces is consistent. Our solution not only allows for evaluation of differences in predictions when specifying an exact initial velocity based on nonlinear theory and its linear approximation, which has been controversial in geophysical practice, but also helps clarify the differences in runup observed during the 2004 and 2005 Sumatran tsunamigenic earthquakes.

DOI: 10.1103/PhysRevLett.97.148501 PACS numbers: 91.30.Nw

The initial value problem (IVP) of the shallow water-wave equations is one of the classic exercises of applied mathematics with literally useful and specific applications in coastal hydrodynamics. For example, when calculating numerically the coastal evolution of tsunamis, the seafloor displacement is used to specify the initial condition for the free surface [1–3]. While analytical solutions for monochromatic waves have been available since the mid-1950s [4], analytical solutions for complex waveforms such as solitary waves for the canonical problem—a uniformly sloping beach connected with a constant-depth segment—were only proposed in the late 1980s [5]. These solutions and their asymptotic approximations have allowed for the validation of numerical codes that predict the shoreline evolution, an otherwise vexing computation [3,6]. Other than for a very mild angle of incidence [7], all existing methods are one-dimensional. For a more complete discussion of tsunami hydrodynamics, refer to Ref. [8].

Recently, Ref. [9] developed a Green's function representation of the solution of an IVP of the nonlinear shallow water-wave (NSW) equations. Linearizing the spatial variable x in the definition of the initial wave, Ref. [9] described a solution for an initial profile with and without initial velocity, employing numerical integration, satisfactory only for small initial amplitudes, when there is nonzero initial velocity. More recently, Ref. [10] used the original transformation presented by Ref. [4], known as Carrier-Greenspan (CG) transformation, to solve an IVP without initial velocity. References [9,10] use the same approximation in the spatial variable to facilitate specification of initial values.

We will suggest a new formulation which appears mathematically more consistent for specifying the IVP for finite initial amplitude with nonzero initial velocity unlike Ref. [9]. We will briefly discuss the formulation of Ref. [9] to clarify its applicability compared with the new solution. Then we will present some geophysical implications using the new formulation.

The two-dimensional NSW equations that describe a propagation problem over the undisturbed water of variable depth $h_0(x) = x$ [Fig. 1] are $[u(h_0 + \eta)]_x + \eta_t = 0$ and $u_t + uu_x + \eta_x = 0$, in nondimensional form. Here $u(x, t)$ and $\eta(x, t)$ are the horizontal depth-averaged velocity and free-surface elevation, respectively. A reference length $\tilde{l}$ is used as a scaling parameter, and dimensionless variables are introduced as $x = \tilde{x}/\tilde{l}$, $h_0 = \tilde{h}_0/(\tilde{l}\tan\beta)$, $\eta = \tilde{\eta}/(\tilde{l}\tan\beta)$, $u = \tilde{u}/(\tilde{g}\tilde{l}\tan\beta)^{1/2}$, and $t = \tilde{t}/(\tilde{l}/(\tilde{g}\tan\beta))^{1/2}$, where β and $\tilde{g}$ are the beach angle from the horizontal and the gravitational acceleration, respectively. Reference [9] used the following transformation:

$$x = \sigma^2 - \eta, \tag{1a}$$

$$t = \lambda + u, \tag{1b}$$

enabling the transform of the NSW equations into

$$(\sigma^2 u)_\sigma + 2\sigma\psi_\lambda = 0, \tag{2a}$$

$$u_\lambda + \frac{1}{2\sigma}\psi_\sigma = 0, \tag{2b}$$

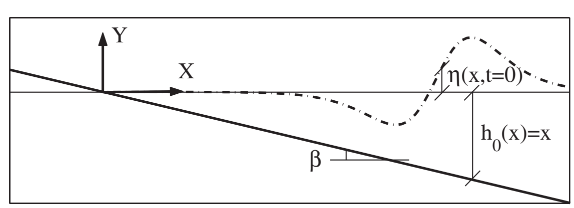

FIG. 1. Definition sketch. Not to scale.

defining a potential function as $\psi = \eta + \frac{1}{2}u^2$. Moreover, Ref. [9] reduced (2a) and (2b) into the following partial differential equation (PDE):

$$4\sigma\psi_{\lambda\lambda} - (\sigma\psi_\sigma)_\sigma = 0. \quad (3)$$

u is defined through (2b) as $u_\lambda = -\psi_\sigma/2\sigma$. However, Ref. [9] did not pursue this transformation.

We consider (3) and general initial conditions in the physical space, i.e., $\eta(x, t = 0) = \eta_0(x)$ and $u(x, t = 0) = u_0(x)$ at $t = 0$. The transformation for the temporal variable (1b), $t = \lambda + u$ results directly into $\lambda = \lambda_0 = -u_0(x)$ when $t = 0$. Then $\eta_0(x)$ and $u_0(x)$ can be converted to the forms $\eta_0(\sigma)$ and $u_0(\sigma)$ using the linearized form of the transformation (1a) $x \approx \sigma^2$. Initial conditions are now defined as $\psi(\sigma, \lambda_0) = \eta_0(\sigma) + \frac{1}{2}u_0^2(\sigma)$ from the definition of the potential function and $\psi_\lambda(\sigma, \lambda_0) = -u_0(\sigma) - \frac{1}{2}\sigma u_{0\sigma}(\sigma)$ from (2a). Given $\eta_0(x)$, the initial velocity can be approximated by $u_0(x) \approx -\eta_0(x)/\sqrt{x}$; we will later discuss the implications of using this linear approximation. The minus sign ensures the initial wave propagates onshore. Thus, given the initial conditions $\psi(\sigma, \lambda_0)$ and $\psi_\lambda(\sigma, \lambda_0)$, then the Hankel transforms give the following solution:

$$\psi(\sigma, \lambda) = 2\int_0^\infty [\psi(\xi, \lambda_0)G_\lambda + \psi_\lambda(\xi, \lambda_0)G]d\xi, \quad (4)$$

where $G(\xi, \sigma, \bar{\lambda}) = \xi \int_0^\infty J_0(\omega\xi)J_0(\omega\sigma)\sin(\frac{1}{2}\omega\bar{\lambda})d\omega$, with $\bar{\lambda} = \lambda - \lambda_0$. The new solution requires one additional integration for the evaluation of $u(\sigma, \lambda)$ from (2b), $u_\lambda = -\psi_\sigma/2\sigma$, allowing direct removal of the singularity at the shoreline using $\lim_{\sigma\to 0}[J_1(\omega\sigma)/\sigma] = \frac{1}{2}\omega$.

Once our solution is obtained in the transform space using (4), it can be converted to the solution in the physical space through (1a) and (1b). Newton-Raphson iterations can be used to obtain a solution at any particular location x^* or any particular time t^* as in Refs. [5,10].

Carrier *et al.* [9] pursued (2a) and (2b) with another potential function, $\varphi_\lambda = \eta + \frac{1}{2}u^2$. They ended up again with the same PDE as (3) for φ; $4\sigma\varphi_{\lambda\lambda} - (\sigma\varphi_\sigma)_\sigma = 0$. Given $u = -\varphi_\sigma/2\sigma$ and their definition of the potential function, the initial conditions are $\varphi(\sigma, 0) = -\int_0^\sigma 2\kappa u(\kappa, 0)d\kappa$ and $\varphi_\lambda(\sigma, 0) = \eta_0(\sigma) + \frac{1}{2}u_0^2(\sigma)$, where $\eta_0(\sigma)$ and $u_0(\sigma)$ are the initial wave and velocity profiles in the transform space, respectively. Reference [9] derived the solution $\varphi(\sigma, \lambda) = 2\int_0^\infty[\varphi(\xi, 0)G_\lambda + \varphi_\lambda(\xi, 0)G]d\xi$, with $G(\xi, \sigma, \lambda) = \xi\int_0^\infty J_0(\omega\xi)J_0(\omega\sigma)\sin(\frac{1}{2}\omega\lambda)d\omega$, which they evaluated in terms of a complete elliptic integral of the first kind, with an essential singularity at $\xi = \frac{1}{2}\lambda - \sigma$.

We differ with the supposition of Ref. [9] that no approximation is needed when specifying their $\varphi(\sigma, 0)$ and $\varphi_\lambda(\sigma, 0)$. Initial conditions may be known in the (x, t) space, but their implementation requires their definition in the (σ, λ) space. Given initial conditions $\eta(x, 0)$ and $u(x, 0)$ at $t = 0$, $\psi(\sigma, \lambda_0)$ and $\psi_\lambda(\sigma, \lambda_0)$ or $\varphi(\sigma, 0)$ and $\varphi_\lambda(\sigma, 0)$ can be defined only if it is assumed $x \approx \sigma^2$. Indeed, Ref. [9] imposed a Gaussian initial wave profile in the following manner:

$$\eta = a\exp[-k(x - x_0)^2] \approx a\exp[-k(\sigma^2 - \sigma_0^2)^2], \quad (5)$$

for $\eta \ll x$, using in essence the approximation of Ref. [5] for the boundary value problem (BVP). While it was claimed [9] that this approximation is made "simply for convenience," it was mathematically necessary to implement initial conditions in the (σ, λ) space [10]. An approximation of the CG transformation was first introduced by Ref. [5], which considered the canonical problem. As argued by Ref. [11], solving the NSW equations for piecewise linear bathymetries is too complex using CG-type transformations. This difficulty was resolved by [5] incorporating the linear theory solution to define a BVP solution for the nonlinear theory and calculating the linear theory solution using the [12] formalism at the transition point. This point is located far off the beach; hence, the nonlinear effects are locally smaller in comparison to closer to the shoreline. Consequently, the solution included a reflection based on the entire wave motion for all times, an unresolved issue in the original and later derivative works using the CG transformation.

In another example, Ref. [9] specified an incident wave from offshore with $u(x, t = 0) \neq 0$, imposed at $\lambda = t = 0$. However, given (1b), λ must be different than zero when $t = 0$ if $u(x, t = 0) \neq 0$. One might claim that when u is small enough, then $t = 0 \approx \lambda$, and this might indeed be a good approximation (as used by Ref. [5] for the BVP solution), but with our new solution, it is entirely unnecessary. No further assumption is needed, and our initial condition is not specified at $t = 0 \approx \lambda$ but instead at $\lambda_0 = -u_0(\sigma)$ corresponding to $t = 0$.

We note that care is needed when comparing the new solution (4) with the Ref. [9] solution. The arguments of the trigonometric functions are proportional to $(\lambda - \lambda_0)$ in (4). One may casually argue that if the initial conditions $\psi(\sigma, \lambda_0)$ and $\psi_\lambda(\sigma, \lambda_0)$ are defined with $\lambda = \lambda_0 = 0$ when $t = 0$ as in Ref. [9], one will anyway obtain a similar solution. One can find λ which will make $t = t^* = 0$. However, the particular λ which will make $t = t^* = 0$ is not $\lambda = 0$, and $\lambda = 0$ does not correspond to $t = t^* = 0$ when $u(x, t = 0) \neq 0$. In short, the Ref. [9] formulation is not consistent with the initial condition in the physical space when there is an initial velocity. A comparison of two cases is presented in Fig. 2. We emphasize that, while the initial condition for $\varphi(\sigma, 0)$ appears in integral form in the Ref. [9] solution, in the new solution the initial conditions are defined with a far simpler expression.

In Fig. 3, we present two examples to allow comparison between our formulation and Ref. [9], using the initial wave form given in (5) with $a = 0.0025$ [Fig. 3(a)] and $a = 0.017$ [Fig. 3(b)] initial waveheights. In both cases, the initial waves have velocity approximated with $u_0(x) \approx -\eta_0(x)/\sqrt{x}$. Reference [9] wrote that, since their solution contains a highly singular function, they were able to perform a satisfactory numerical computation for a small

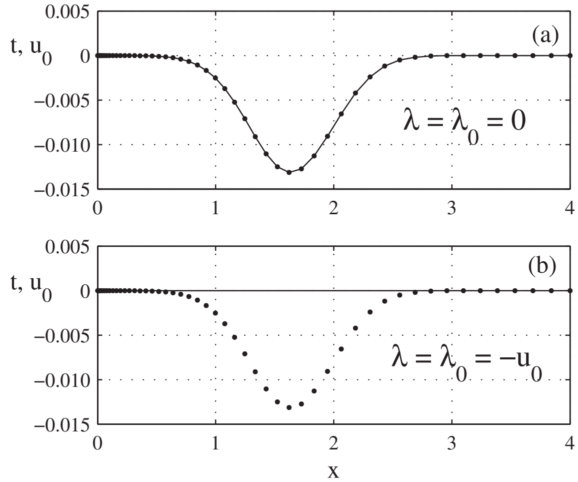

FIG. 2. Nonlinear evolution of the initial waveforms defined in (5) with initial velocity with $a = 0.017$, $x_0 = 1.69$, and $k = 4.0$. The solid line and the dots represent time and velocity, respectively. (a) Reference [9]-like solution: Taking $\lambda = \lambda_0 = 0$ in the present solution produces $t = u_0$. $t \neq 0$ unlike Ref. [9] claims. (b) Proposed solution: $t = 0$ when $\lambda = \lambda_0 = -u_0$ required by the transformation for the temporal variable (1b).

value of the initial wave amplitude. Unlike Ref. [9], we do not have any difficulty in resolving the entire flow field with (4), even for large values of initial wave amplitude [Fig. 3(b)]. We believe a major hindrance with Ref. [9] is the implementation of one of the initial conditions in the integral form. In the same figure, we present the time evolution of a wave with $a = 0.017$ without an initial velocity [Fig. 3(c)] and compare shoreline motions for both cases [Fig. 3(d)], i.e., with and without initial velocity. There is a difference in the maximum runup of a factor of 2. This difference suggests strong implications in the modeling of landslide-generated waves. Current practice ignores the time dependence of the generation process, and this may result in a substantial difference in the predictions of the maximum runup if the appropriate initial velocity is not specified.

Carrier *et al.* [9] observed a small outgoing "noise" and attributed it to the mismatch caused by the linear approximation for $u_0 \approx -\eta_0/\sqrt{x}$. While one can approximate the velocity field $u_0 \approx -\eta_0/\sqrt{x}$ [13], one does not need to and instead can use correct nonlinear value $u_0 = -(2\sqrt{x + \eta_0} - 2\sqrt{x}) = -2\sigma + 2\sqrt{\sigma^2 - \eta_0}$. However, it may not be always possible to perform satisfactory analytical computations for each time step. Yet, even using the "full" nonlinear value for the initial wave as with our methodology, the outgoing wave does not disappear [Fig. 3(e)].

An unresolved question when initializing geophysical tsunami forecasting models [1] remains the specification of the initial velocity. Given that the fault rupture is instantaneous compared to the free-surface wave generation [14,15], the standard practice in numerical modeling has been to transfer the inferred seafloor displacement to the

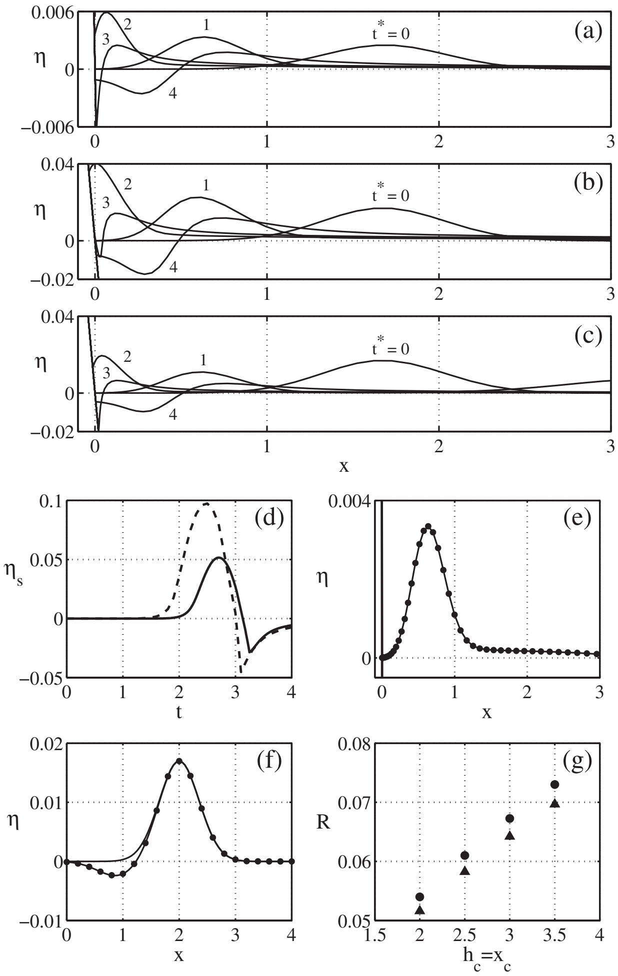

FIG. 3. Nonlinear evolution of the initial waveforms defined in (5) with initial velocity. Cases presented are (a) $a = 0.0025$, $x_0 = 1.69$, and $k = 4.0$ and (b) $a = 0.017$, $x_0 = 1.69$, and $k = 4.0$. Initial wave velocities are defined using $u_0 \approx -\eta_0/\sqrt{x}$. (c) Nonlinear evolution of an initial waveform given in inset (b) without initial velocity. (d) Comparison of the shoreline motions for the cases presented in insets (b) and (c), i.e., with (dashed line) and without (solid line) initial velocity. (e) Nonlinear evolution at the time $t^* = 1$ for the case presented in inset (a). The solid line and the dots represent cases where initial velocities are defined using a linear approximation and "full" nonlinear value, respectively. (f) A Gaussian ($a = 0.017$, $x_c = 2.00$, and $k = 4.0$) and a leading-depression N wave (previous Gaussian fronted by a depression with $a = 0.0025$, $x_d = 0.9$, and $k = 4.0$) initial forms having the same maximum amplitudes at the same location. (g) The maximum runup values for a Gaussian (triangles) and a leading-depression N wave (circles) for various initial wave locations.

free surface and form a well-posed IVP with zero velocity. Further, current state-of-the-art forecasting methodology [16] employs superposition. Precomputed farfield amplitude and velocity fields for "unit" tsunamigenic faults for specific scenario events are linearly combined to provide initial conditions for local nonlinear inundation models at high resolution near target coastlines. Given that only a handful of free-field tsunami amplitude measurements exist with no associated initial velocity data, there is controversy as to the adequacy of the linear superposition of the velocity field.

Next, we are now able to assess the effect of the initial location of wave crest on a maximum runup. We introduce a Gaussian and a leading-depression N wave having the maximum amplitudes at the same location as in Fig. 3(f). We then vary this initial location and present its effect over the maximum runup in Fig. 3(g). The runup is seen to vary almost linearly with the depth. This confirms the speculation (see the maps attributed to Synolakis and Arcas in Ref. [17]) for the observed substantial differences in inundation locally between the $M_s \approx 9.3$ December 26, 2004 and the $M_s \approx 8.7$ March 28, 2005 Sumatra events, which could not be attributed to the differences in the seismic moment. While the epicenter of the 2004 event was further off the shoreline in deeper water, the observed runup differences cannot be attributed solely to the differences in depth, at least as suggested from Fig. 3(g). The two offshore islands of Nias and Simeulue, underneath which most of the seismic deformation of the 2005 event is believed to have occurred, limited the volume of fluid available for tsunamigenesis. Further, Fig. 3(g), which presents NSW results, confirms the observation of Ref. [18] based on linear theory, that leading-depression N waves run up further than leading-elevation N waves of the same initial height. Note that even the presence of the small depression wave ahead of the Gaussian affects the runup to first order.

As recent satellite altimeter data have suggested [19], the midocean steepness of the 2004 tsunami was less than 10^{-5}. Once the wave arrives at the toe of the beach—typically of an extent of the order of one wavelength or less—dispersion is far less important than nonlinearity in the subsequent nearshore evolution; hence, the NSW theory presented here is applicable. Nonlinear dispersive theory is necessary only when examining steep gravity waves [20,21], which mercifully are not encountered in the context of tsunami hydrodynamics in deep water but otherwise at abrupt transitions in depth.

Finally, there are two basic differences in the existing BVP and IVP solutions of NSW equations. First, the IVP solutions of Refs. [9,10] do not solve the canonical problem and need to specify Gaussian or N waves directly on the sloping beach and then determine their evolution. Clearly, the representation of an initial wave by approximating the spatial variable x may introduce distortions in the transform spatial variable σ, given the necessary linearization. In fact, Ref. [9] wrote that their specification is valid for $\eta \ll x$, some distance offshore. While this does not appear at first as a drawback, small-amplitude solitary waves have long wavelengths that may often extend over the entire length of the sloping beach. The approximations thus introduced may not be uniformly valid. Second, the physical problem involves a wave coming from offshore and then evolving over coastal topography. For example, in laboratory realizations, solitary waves coming from infinity evolve first over a constant depth and then change significantly as the waveform propagates over the sloping beach; see Fig. 9(a) of Ref. [5]. In the BVP analysis, this difficulty is avoided.

In summary, we described a new method for solving the IVP of the NSW equations that is consistent with the definition of the initial condition in the physical and transform spaces with nonzero initial velocity. The proposed formulation appears simpler than earlier work and also appears to extend it beyond small initial waveheights with initial velocities. The differences in runup suggested from linear theory between leading-elevation and leading-depression waves persist even in our consistent nonlinear formulation of the IVP.

Our comparison of solutions with initial velocity specified through its exact nonlinear value and its linear approximation suggests that the latter produces a result almost indistinguishable from the former. This appears to be one of the reasons that current operational tsunami forecast models appear to model geophysical reality better than otherwise expected [16], given the awkwardness of linearly superposing unitary profiles in a nonlinear model.

[1] V. Titov *et al.*, Science **309**, 2045 (2005).
[2] V. V. Titov and C. E. Synolakis, Geophys. Res. Lett. **24**, 1315 (1997).
[3] C. E. Synolakis *et al.*, Science **278**, 598 (1997).
[4] G. F. Carrier and H. P. Greenspan, J. Fluid Mech. **4**, 97 (1958).
[5] C. E. Synolakis, J. Fluid Mech. **185**, 523 (1987).
[6] P. L.-F. Liu, C. E. Synolakis, and H. H. Yeh, J. Fluid Mech. **229**, 675 (1991).
[7] M. Brocchini and D. H. Peregrine, J. Fluid Mech. **317**, 241 (1996).
[8] C. E. Synolakis and E. N. Bernard, Phil. Trans. R. Soc. A **364**, 2231 (2006).
[9] G. F. Carrier, T. T. Wu, and H. Yeh, J. Fluid Mech. **475**, 79 (2003).
[10] U. Kânoğlu, J. Fluid Mech. **513**, 363 (2004).
[11] E. Pelinovsky, Appl. Anal. **57**, 63 (1995).
[12] J. B. Keller and H. B. Keller, ONR Research Report No. NONR-3828(00), 1964.
[13] V. V. Titov and C. E. Synolakis, J. Waterw. Port Coast. Ocean Eng. **121**, 308 (1995).
[14] A. Ben-Menahem and M. Rosenman, J. Geophys. Res. **77**, 3097 (1972).
[15] J. L. Hammack, Ph.D. thesis, Caltech, 1972.
[16] V. V. Titov *et al.*, Nat. Hazards **35**, 41 (2005).
[17] R. A. Kerr, Science **308**, 341 (2005).
[18] S. Tadepalli and C. E. Synolakis, Phys. Rev. Lett. **77**, 2141 (1996).
[19] E. L. Geist, V. V. Titov, and C. E. Synolakis, Sci. Am. **294**, No. 1, 56 (2006).
[20] J. L. Hammack and H. Segur, J. Fluid Mech. **65**, 289 (1974).
[21] A. S. Fokas and Q. M. Liu, Phys. Rev. Lett. **77**, 2347 (1996).