Coarse Grained Scale of Turbulent Mixtures

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What is the physical length scale which supports the concentration content in a stirred mixture? Among the length scales familiar in stirred mixtures is the dissipation scale which equilibrates substrate deformation and diffusive smearing rates. That scale is a decreasing function of the deformation rate and is, thus, a decreasing function of the Reynolds number in turbulent flows. Experiments show that the mixture concentration content is defined on a support whose elementary brick $\eta = L\mathrm{Sc}^{-2/5}$ is much larger. It scales like the stirring scale L, depends on the Schmidt number Sc, and is independent of the Reynolds number. The above law is supported by measurements covering two decades in L and three decades in Sc. We suggest that this scale results from the aggregation of bundles of elementary stretched scalar sheets merging under large-scale substrate deformation.

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There are several physical length scales naturally involved in a stirred scalar mixture. Suppose that a blob of dyed fluid is deposed in a medium of the same fluid. First, in the absence of stirring, that is, in the pure diffusion limit on a still substrate, an obvious spatial scale is the initial size s_0 of the blob. The concentration in the blob has appreciably decayed from its initial value when the blob has been smeared by diffusion, that is, when its current radius $\sqrt{Dt}$ is appreciably larger than s_0, and this happens when $t \gg s_0^2/D$ according to the usual estimate if D is the scalar diffusion coefficient. Then, with stirring, the picture is substantially altered. Diffusive smearing is hastened in the presence of stretching because the scalar gradient is constantly steepened by compression. Stretching motions are accompanied, in incompressible fluids, by compressive motions. The scalar concentration is close to uniform along the stretching directions and varies rapidly along the compressive one, thus forming a sheetslike topology [1,2]. This, in turn, also sets a characteristic equilibrium scale in the field itself. Let $s(t)$ be the distance between two material points aligned with the compressive direction. The reduction of the sheet concentration profile width goes on until the rate of compression $\gamma = -d \log s(t)/dt$ balances the rate diffusive broadening on its current size $D/s(t)^2$, that is,

$$\frac{D}{s^2} \sim -\frac{1}{s}\frac{ds}{dt} = \gamma, \tag{1}$$

and this equilibrium is reached at the *mixing time* t_s of the sheet. If the rate of compression γ is constant in time, then $s(t) = s_0 e^{-\gamma t}$ and $t_s \sim \frac{1}{2\gamma} \ln(\gamma s_0^2/D)$. The typical width of the scalar gradient which "dissipates" the scalar differences equals

$$s(t_s) \sim s_0 e^{-\gamma t_s} = \sqrt{\frac{D}{\gamma}}, \tag{2}$$

usually called the Batchelor scale [3]. When stretching of material surfaces is algebraic in time like $s(t) \sim s_0(\sigma t)^{-\beta}$, with β some positive exponent, the rate of compression diminishes in time like $\gamma = \beta/t$. The concentration gradient at the mixing time spans over a typical distance of the order of [4–6]

$$s(t_s) \sim s_0(\sigma t_s)^{-\beta} = s_0 \mathrm{Pe}^{-\beta/(2\beta+1)}, \tag{3}$$

with the Péclet number $\mathrm{Pe} = \sigma s_0^2/D \gg 1$. The same procedure applies to a compression rate $\gamma(s)$ constant in time but which depends on the separation distance s itself. Choosing, for instance, $\gamma(s) \sim (\epsilon s)^{1/3}/s$ as suitable for high Reynolds number turbulent flows [7], the corresponding "dissipation scale" equilibrating substrate compression and diffusive broadening is

$$\left(\frac{D^3}{\epsilon}\right)^{1/4}, \tag{4}$$

a scale usually referred to as the Corrsin-Obukhov [8,9] length scale.

This local equilibrium paradigm applies beyond the scalar mixing context and is successful at describing the aspect ratio of drops immersed in a nonmiscible sheared substrate [10] or the maximal size of drops and bubbles in a turbulent flow [11,12], by an appropriate balance between hydrodynamics stresses and capillary restoration forces at the scale of the drops themselves.

The global coarse grained scale.—All of the above scales describe the size of an *isolated* object arising from the *local* balance between a time characteristic of the motions in the underlying substrate and a diffusion time. However, evidences have been provided suggesting that the global concentration content of a stirred mixture results from the merging of *nearby* objects, namely, scalar sheets, which interact though a *random aggregation* process [13]. This phenomenon has an incidence on some geometrical

facets of the mixture. In particular, we show below that the concentration field C of the mixture is smooth on a scale which is much larger than the typical scale of its gradient estimated from the above length scales and that this "coarse grained scale" directly reflects the aggregation construction of the concentration field.

Experimentally, the scalar is injected (at scale d) in various sustained, large-scale advected turbulent shear flows (integral scale L) in which it disperses and mixes [14,15]. Concentration is recorded at one point downstream of the source providing a signal $C(x)$, where distances x and time t are related to each other by Taylor hypothesis. Some measurements have also been made on two-dimensional cuts through the field using a fluorescent dye (Fig. 1). Table I summarizes the range of Reynolds and Schmidt numbers and the injection ratio used (see also Fig. 2).

We denote $V_>(r)$ the variance of the concentration field coarse grained at scale r. If $F(k)$ is the field spectrum [15,16], then

$$V_>(r) = \int_{2\pi/L}^{2\pi/r} F(k)dk. \qquad (5)$$

Equivalently, $V_>(r)$ can be computed by filtering the original concentration field $C(x)$ with a top hat window of width
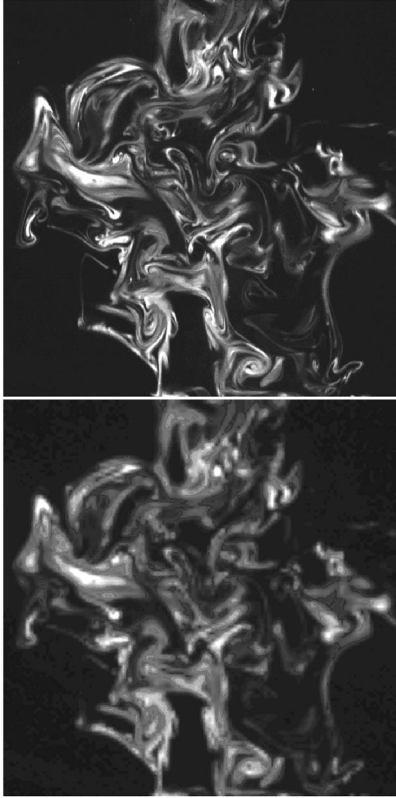

FIG. 1. Coarse graining operation on a two-dimensional cut through a scalar field (Sc = 2000). The original field (top) has been coarse grained up to the scale η (bottom).

r defining a coarse grained field $C_r = C(x) \otimes H_r(x)$, with $H_r(x)$ equal to $1/r$ for $0 < x < r$ and equal to 0 for $x > r$. Then

$$V_>(r) = \langle (C_r - \langle C_r \rangle)^2 \rangle. \qquad (6)$$

We have used the above definition to compute $V_>(r)$ from the original concentration signals. The operation can be seen visually in Fig. 1 on a two-dimensional cut though the field and the dependence of $V_>(r)$ on r is shown in Fig. 2 for different injection and stirring conditions. The coarse grained variance presents an inverted S shape with the following features: It is independent of the Reynolds number Re and is shifted by varying the diffusivity of the scalar measured by the Schmidt number Sc. It is, however, independent of the scalar injection scale d. Half of the variance has been erased by coarse graining the field up to the scale $r = \eta$, which coincides approximately with the location of the S-shape inflection point. All measurements for different scalars and flows in Fig. 3 are consistent with

$$\eta = L\mathrm{Sc}^{-2/5}. \qquad (7)$$

The *coarse grained scale* η depends solely on the large, stirring scale of the flow and on the diffusional properties of the scalar. We explain below its origin.

Aggregation of a bundle of sheets.—Consider no more an isolated scalar sheet being stretched, but instead a bundle of parallel sheets in the process of merging into each other under the action of a large-scale stretching rate. This process occurs permanently in a stirred mixture and gives rise to the self-convolution construction of the concentration distribution [13]. We will consider that the sheets are stretched in two directions parallel to their plane under the action of an elongation rate σ, identical in the two directions, constant in time, and uniform over a scale L resulting in a global compression along the x axis so that

$$s(t) = \frac{s_0}{1 + (\sigma t)^2}. \qquad (8)$$

This law applies to sustained shear flows such as shear layers [4] or jets like in the present experiments. We choose a simple initial scalar field $C(x, t = 0)$ consisting in a bundle of parallel sheets, each separated from their immediate neighbors by a distance $2\pi/k_0$, and piled up over a distance of the order of L. The sheets are compressed in the direction x parallel to their transverse size. We first disregard the distribution of the concentration levels between the sheets. A functional form for $C(x, t = 0)$ having the required features is

TABLE I. Range of experimental parameters. The flow rms velocity is u', kinematic viscosity is ν, and D is the scalar diffusivity.

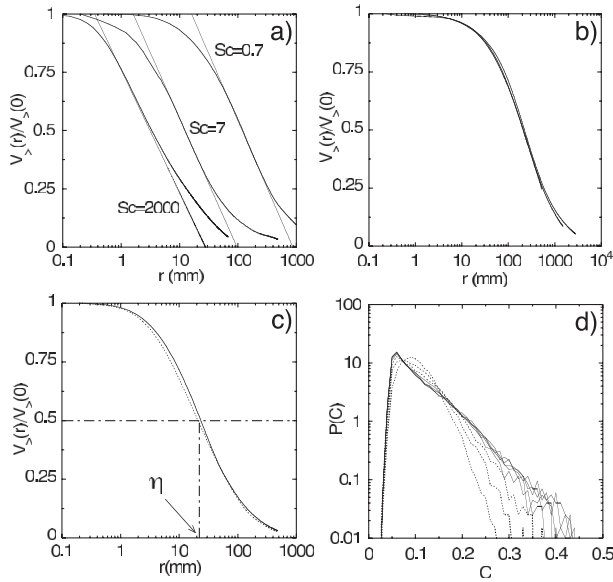| Re = $u'L/\nu$	Sc = ν/D	d/L
10^3–10^7	0.7–2000	0.05–1

FIG. 2. The coarse grained variance of the concentration field $V_>(r)$ as a function of the coarse grained distance r for different scalars and stirring conditions with $L = 6$ cm. (a) $V_>(r)$ versus r for three Schmidt numbers at a fixed location downstream of the scalar source. Half of the variance of the field has been erased by coarse graining at the scale $r = \eta$, which depends on Sc. (b) At a fixed location $x/d = 20$ downstream of a $d = 3.3$ cm source in air (Sc $= 0.7$) for $u = 3$ m/s (Re $= 9000$), $u = 6.2$ m/s (Re $= 19\,000$), and $u = 11.6$ m/s (Re $= 35\,000$). The corresponding Corrsin-Obukhov length is $\eta_{CO} = L\mathrm{Re}^{-3/4}\mathrm{Sc}^{-3/4} \approx 0.1$ mm. (c) At a fixed location $x/d = 12.5$ and Sc $= 7$ for Re $= 6000$ and two different injection diameters $d = 6$ mm (dotted line) and $d = 10$ mm (continuous line). The corresponding Batchelor length is $\eta_B = L\mathrm{Re}^{-3/4}\mathrm{Sc}^{-1/2} \approx 0.03$ mm. (d) Change of the concentration distribution $P(C)$ as the coarse graining scale is increased for $x/d = 10$, Sc $= 0.7$, and Re $= 45\,000$: $r = 2$–256 mm by factors of 2. There are 5 curves with $r < \eta$ and 3 with $r > \eta$ (dashed).

$$C(x, t = 0) = 1 + \cos(k_0 x), \qquad (9)$$

for $-L/2 < x < L/2$ and equal to 0 elsewhere. The convection-diffusion equation for $C(x, t)$

$$\partial_t C - \gamma(t)x\partial_x C = D\partial_x^2 C \qquad (10)$$

is a pure diffusion equation in the $\{\xi, \tau\}$ variables thanks to a suitable transformation of space and time [4,5,13,17]

$$\tau = D \int_0^t \frac{dt'}{s(t')^2} \quad \text{and} \quad \xi = \frac{x}{s(t)} \qquad (11)$$

and is solved to give the concentration field at any time as

$$C(x, t) = 1 + \cos(\xi)e^{-\tau}, \qquad (12)$$

with

$$\xi = k_0 x(1 + (\sigma t)^2), \qquad \tau = Dk_0^2 t(1 + \tfrac{2}{3}(\sigma t)^2 + \tfrac{1}{5}(\sigma t)^4). \qquad (13)$$
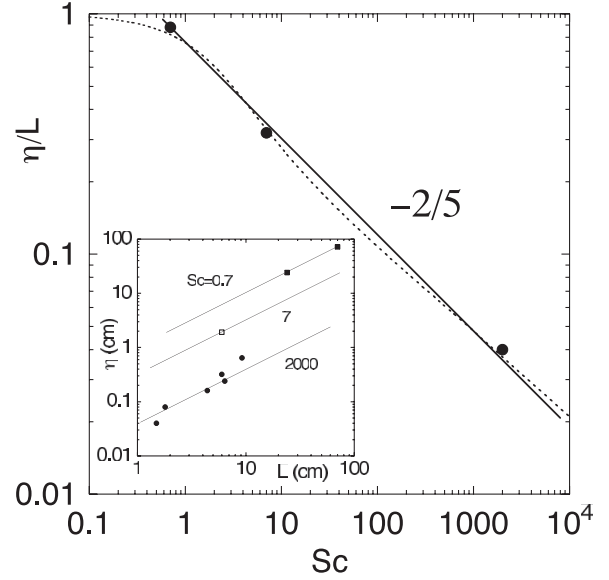
FIG. 3. Coarse grained scale η versus integral scale L for various flows, stirring conditions, and Schmidt numbers. The dotted line is the exact solution from Eq. (13) with $\tau = 0.15$. The continuous line is the asymptotic trend for Pe $\gg 1$ in Eq. (7). For Pe $\ll 1$, $\eta = L$.

The time needed to complete sheet coalescence in the bundle is the time required to make concentration modulations small compared to unity, that is, to make the factor $e^{-\tau}$ appreciably small [see Eq. (12)]. This occurs from the instant of time making τ of order unity, defining the mixing or coalescence time as

$$t_s \sim \frac{1}{\sigma}\mathrm{Pe}^{1/5}, \quad \text{with} \quad \mathrm{Pe} = \frac{\sigma}{Dk_0^2} \gg 1. \qquad (14)$$

At that instant of time, the concentration in the bundle of merged sheets is close to uniform; the modulations have been essentially erased (Fig. 4). The transverse size of the bundle, initially equal to L, has decreased accordingly. It has been compressed by an amount $1 + (\sigma t_s)^2 \approx \mathrm{Pe}^{2/5}$ [see Eq. (8)]. The transverse size of the bundle defining a region of close-to-uniform concentration is thus
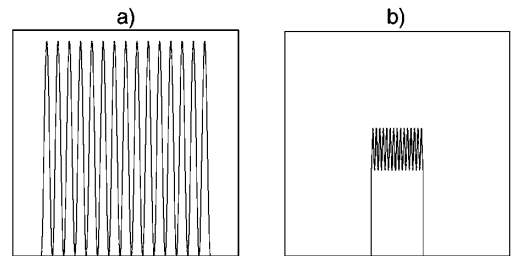

FIG. 4. (a) Concentration profile of a bundle of sheets piled up initially between $-L/2 < x < L/2$ and evolving according to Eq. (12). (b) At $t \approx t_s \sim \frac{1}{\sigma}\mathrm{Pe}^{1/5}$, concentration modulations in the bundle have been substantially reduced while the size of the bundle has been shrunk by a factor $1 + (\sigma t_s)^2$, thus defining η.

$$\eta = L\mathrm{Pe}^{-2/5}. \qquad (15)$$

For $\mathrm{Pe} \ll 1$, the mixing time t_s is of order s_0^2/D and $\eta = L/(1 + \mathrm{Pe}^2) \approx L$, independent of Pe. If k_0^{-1} is further set to $s_0 \sim \sqrt{\nu/\sigma}$, the Taylor microscale of the flow [18], as suggested by the discussion in Ref. [14] explaining how this choice is the only one compatible with the independence of the scalar concentration distribution on the Reynolds number (see also Fig. 2), then $\mathrm{Pe} = \sigma/Dk_0^2 = \nu/D = \mathrm{Sc}$, and η coincides with the anticipated value given in Eq. (7).

This result is not contingent to the particular choice made for the initial concentration profile in Eq. (9). If, instead, one considers a set of adjacent sheets of width s_0 with distributed initial concentrations, the spectrum of such a concentration field is computed exactly as $F(k) = (\sigma_0^2/\pi s_0 k^2)(1 - \cos(ks_0))$ if $\sigma_0^2 = V_>(0)$ is its initial variance. It can be shown [19] from the diffusive relaxation of the spectrum that the variance σ^2 at any later time is

$$\frac{\sigma^2}{\sigma_0^2} = \frac{1}{\sqrt{\pi}\psi}(e^{-\psi^2} - 1) + \mathrm{erf}(\psi), \qquad (16)$$

with $\psi = 1/\sqrt{8\tau}$. The variance has been reduced by a factor close to $1/2$ when $\psi = 1$, which is precisely the criterion used to define η. The analysis can be conducted with an initial distribution of sheet thicknesses [20] as well.

The above mechanism showing the existence in the scalar field of a scale of the order of the large, stirring scale is reminiscent of the ramp-cliff-plateau structures observed long ago in shear flows. There, L-wide regions of nearly uniform concentration are separated by steep cliffs absorbing a concentration difference of the order of the mean [21–24].

As opposed to local balances setting the fine-scale gradient size of elementary sheets, this new scale results from a nonlocal, fusion mechanism itself consistent with the construction mechanism of the concentration distribution. Indeed, the concentration distribution of randomly stirred mixtures has been shown to evolve, in some cases, through a self-convolution process: As soon as the mixture is stirred in a more or less random manner, its concentration content evolves by diffusive coalescence of the concentration levels between nearby elements in an additive fashion and occurs at random among the levels available in the current distribution, which is found to be very well represented by this process [13]. This fusion mechanism between nearby sheets is, also, precisely the one giving rise to the coarse grained scale η.

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