Coexistence of Josephson Oscillations and a Novel Self-Trapping Regime in Optical Waveguide Arrays

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(Received 20 April 2006; published 4 October 2006)

Considering the coherent nonlinear dynamics between two weakly linked optical waveguide arrays, we find the first example of *coexistence* of Josephson oscillations with a novel self-trapping regime. This macroscopic bistability is explained by proving analytically the simultaneous existence of symmetric, antisymmetric, and asymmetric stationary solutions of the associated nonlinear Schrödinger equation. The effect is illustrated and confirmed by numerical simulations. This property allows us to conceive an optical switch based on the variation of the refractive index of the linking central waveguide.

DOI: 10.1103/PhysRevLett.97.143902 PACS numbers: 42.65.Wi, 05.45.−a

Introduction.—Since its prediction in 1962 [1], immediately followed by an experimental verification [2], the Josephson effect has been widely applied in various branches of physics. It is a macroscopic quantum tunneling effect, originally discovered in superconducting junctions, caused by the global phase coherence between electrons in the different layers. Similar *Josephson oscillations* have been discovered in liquid helium [3,4] and in double layer quantum Hall systems [5,6].

The first experimental observation of a bosonic Josephson junction has been made for a Bose-Einstein condensate embedded in an optical lattice [7], and very recently it was realized for a macroscopic double well potential [8]. The difference with the ordinary Josephson junction behavior is that the oscillations of population imbalance are suppressed for high imbalance values and a self-trapping regime emerges [9,10].

The nonlinear dynamics of bosonic junctions, described by the Gross-Pitaevskii equation (GPE) [11], is usually mapped to a simpler system characterized by 2 degrees of freedom (population imbalance and phase difference), while the nonlinear properties of the wave function within the single well are neglected. In this approach, the symmetric and antisymmetric stationary solutions of GPE are used as a basis to build a global wave function [12,13]. This description allows one to show that for higher nonlinearities the symmetric solutions become unstable and degenerate to an asymmetric stationary (approximate) solution of GPE corresponding to a new self-trapping regime [14,15].

The optical realization of a bosonic junction had been theoretically proposed much earlier by Jensen [16], who considered light power oscillations in two coupled nonlinear waveguides, which actually realizes Josephson oscillations in the *spatial domain* where the governing model is the nonlinear Schrödinger equation (NLS).

Considering the two weakly linked optical waveguide arrays in Fig. 1 with light injected in one array, we discover, in a wide range of input intensity, that light can either remain trapped in this array or swing periodically from right to left and back as shown by the intensity plot in Fig. 2. The switching from one state to the other is triggered by a slight local variation of the refractive index of the central linking waveguide. The *coexistence* of oscillatory and self-trapping regimes corresponds to the simultaneous presence of Josephson oscillations and an asymmetric solution of the NLS.

Our result differs from known behaviors of bosonic Josephson junctions, where the presence of oscillatory or self-trapping regimes is uniquely determined by the parameters of the system. The resulting switching property is likely to have a straightforward experimental realization in waveguide arrays, which constitute truly one-dimensional systems and are particularly convenient for observation of nonlinear effects. Indeed, many experimental realizations have revealed nice nonlinear properties, such as soliton generation and guiding; see, e.g., [17–23].

Model and numerical simulations.—An array of adjacent waveguides coupled by power exchange is modeled

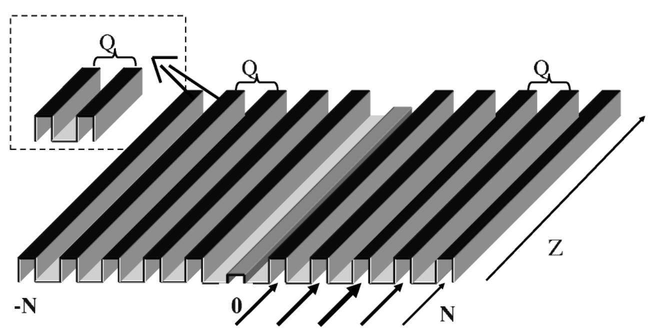

FIG. 1. The two weakly linked waveguide arrays: The refractive index of the central waveguide is smaller than the indices of the waveguides in the two arrays. The inset displays the elementary cell with the coupling constant Q.

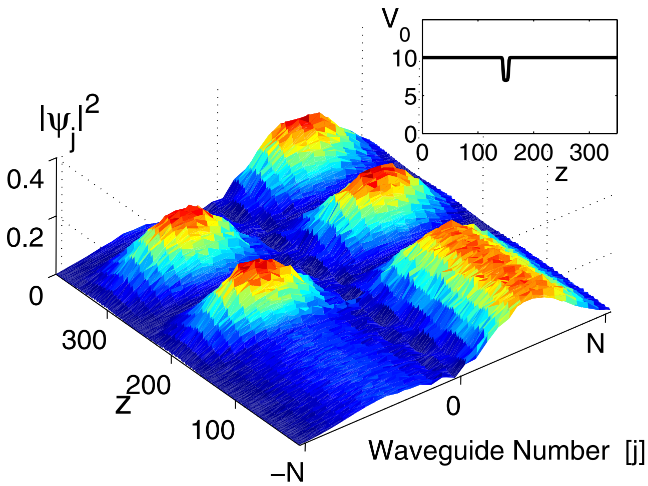

FIG. 2 (color online). Numerical simulation of the DNLS Eq. (1) with initial condition (3) and parameters (2). By a slight local variation at $z = 150$ of the refractive index of the central waveguide, represented in the inset in terms of the relative barrier height V_0, the regime switches from self-trapping to Josephson oscillations. The injected total power is $P_t = \sum |\psi_j|^2 = 1.44$.

by the discrete nonlinear Schrödinger equation (DNLS) [24,25], which reads

$$i\frac{\partial \psi_j}{\partial z} + \frac{\omega}{c}(n_j - n)\psi_j + Q(\psi_{j+1} + \psi_{j-1} - 2\psi_j) + |\psi_j|^2\psi_j = 0, \quad (1)$$

where waveguide discrete positions are labeled by the index $j\,(-N \le j \le N)$, and the complex field ψ_j results from the projection of the electric field envelope on the eigenmode of the individual waveguide. It is normalized to a unit on-site nonlinearity. The linear refractive index n_j is set to n for all $j \neq 0$ and to $n_0 < n$ for $j = 0$. The coupling constant between two adjacent waveguides is Q, and ω and c are the light frequency and velocity, respectively. Vanishing boundary conditions $\psi_{N+1} = \psi_{-N-1} = 0$ model a strongly evanescent field outside the waveguides.

Written for two waveguides, the above equation reduces to the one considered by Jensen [16] for the elementary cell in Fig. 1. In that case, for a beam of small intensity, light tunnels from one waveguide to the other and then back, inducing Josephson oscillations [9,10]. Increasing the input intensity, above some critical value, light becomes self-trapped in one waveguide, a behavior characteristic of *bosonic junctions*.

We demonstrate now by numerical simulations of model (1) that, for the device in Fig. 1, the two regimes, namely, Josephson oscillations and self-trapping, *coexist* for a given injected beam intensity and given parameter values, the switch from one state to the other being obtained by a tiny local variation of the refractive index of the central waveguide.

Let us choose the following values for the parameters in Eq. (1):

$$N = 14, \qquad Q = 16, \qquad \omega(n - n_0)/c \equiv V_0 = 20, \quad (2)$$

together with the following input light envelope:

$$\psi_j(0) = 0.4\sin[(N + 1 - j)/5.5], \qquad j = 1, \cdots N,$$
$$\psi_j(0) = 0.2\sin[(N + 1 + j)/5.5], \qquad j = -N, \cdots 0, \quad (3)$$

which represents a beam mostly sent into the right waveguide array. The result is displayed in Fig. 2. While the relative refractive index V_0 of the central waveguide is kept constant, the power injected initially into the right part of the array remains self-trapped. At $z = 150$, a local variation of V_0, drawn in the inset in Fig. 2, makes the self-trapping state bifurcate to a regime of Josephson oscillations, which then remains stable and demonstrates a novel *bistability* of the coupled array.

Theory.—We shall now interpret these results in terms of the continuum limit of model (1). Considering $1/\sqrt{Q}$ as being a virtual grid spacing, we may represent $\psi_j(z)$ by the function $\psi(x, z)$ in the continuous variable $x = j/\sqrt{Q}$. As a result, the DNLS model (1) maps to the NLS equation

$$i\frac{\partial \psi}{\partial z} + \frac{\partial^2 \psi}{\partial x^2} - V(x)\psi + |\psi|^2\psi = 0, \quad (4)$$

where $V(x)$ is a double well potential with a width $2L = (2N + 2)/\sqrt{Q}$, represented in Fig. 3. The potential barrier has height V_0 and width $2l = 1/\sqrt{Q}$, and we assume, for technical simplification, that the Schrödinger equation is

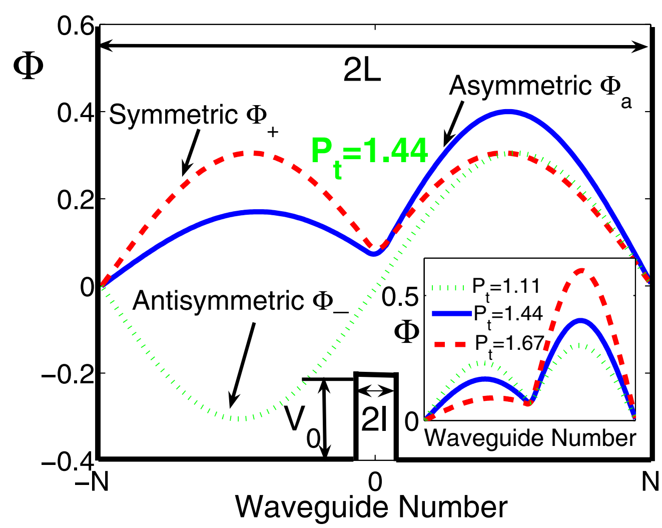

FIG. 3 (color online). Plot of the double well square potential for the continuous model (4): $2L$ is the well width; V_0 and $2l$ are barrier height and width, respectively. The curves are the plots of different types of solutions obtained for the total power $P_t = 1.44$. The inset shows the form of the asymmetric solution for different values of the total power.

linear inside the barrier. Numerical simulations are performed with a fully nonlinear array.

The stationary solution of (4) is sought as $\psi(z,x) = \Phi(x)\exp(-i\beta z)$, with a real-valued function $\Phi(x)$ found in terms of Jacobi elliptic functions [26]

$$\begin{aligned} \Phi &= B\,\mathrm{cn}(\gamma_B(x+L) - \mathbb{K}(k_B), k_B) \qquad (-L < x < -l),\\ \Phi &= a e^{\lambda x} + b e^{-\lambda x}, \qquad \lambda^2 = V_0 - \beta \qquad (-l < x < l),\\ \Phi &= A\,\mathrm{cn}(\gamma_A(x-L) + \mathbb{K}(k_A), k_A) \qquad (l < x < L), \end{aligned} \tag{5}$$

with the parameters given in terms of the amplitudes by

$$\gamma_A = \sqrt{A^2+\beta}, \qquad \gamma_B = \sqrt{B^2+\beta}, \tag{6}$$

$$k_A^2 = \frac{A^2}{2(A^2+\beta)}, \qquad k_B^2 = \frac{B^2}{2(B^2+\beta)}, \tag{7}$$

where β is an eigenmode of a single waveguide ($\beta < V_0$) and where $\mathbb{K}$ denotes the complete elliptic integral of the first kind. By construction, the above expressions verify the vanishing boundary values in $x = \pm L$.

The solutions are then given in terms of five parameters (A, B, a, b, and β), four of which are determined by the continuity conditions in $x = \pm l$. Thus, the conserved total injected power $P_t = \int |\psi|^2 dx$ completely determines the solutions. Another useful conserved quantity is the total energy E given by

$$E = \int\left(\left|\frac{\partial\psi}{\partial x}\right|^2 + V(x)|\psi|^2 - \frac{|\psi|^4}{2}\right)dx. \tag{8}$$

In the weakly nonlinear limit (small P_t), the solutions are symmetric (odd or even). The even solution $\Phi_+(x)$ corresponds to $A = B$ in (5) when the solution in the barrier region is $2a\cosh(\lambda x)$. The odd solution $\Phi_-(x)$ corresponds to $A = -B$ with central solution $2a\sinh(\lambda x)$. For higher powers, namely, above a threshold value, an asymmetric solution $\Phi_a(x)$ also exists for which $A \neq \pm B$. These analytical solutions are represented in Fig. 3.

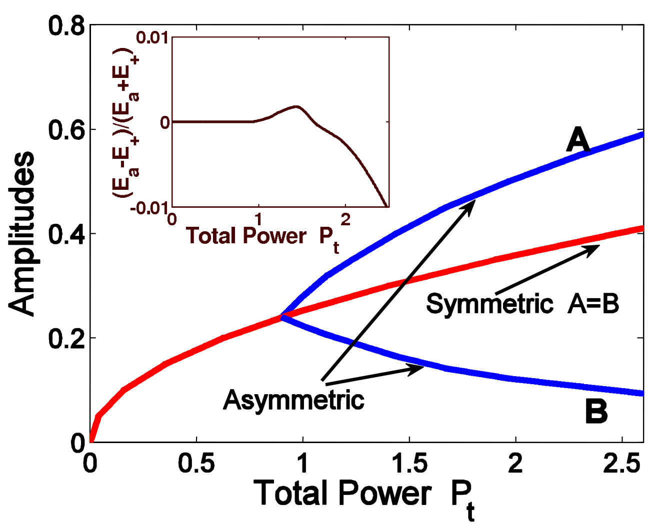

FIG. 4 (color online). Dependence of the amplitudes [maximum values A and B of the expressions (5)] of the symmetric and asymmetric solutions on the total power (the amplitudes of the odd and even symmetric solutions almost superpose). The inset displays the relative energy difference of the symmetric (Φ_+) and asymmetric (Φ_a) solutions in terms of the total power.

To plot the solutions, we stick with the parameter values which follow from (2): The width of the rectangular double well potential is $2L = 7.5$, the barrier width is $2l = 0.25$, and its height is $V_0 = 20$. We derive the complete set of solutions (5) and display the dependence of their amplitudes on the total power $P_t = \int |\psi|^2 dx$ in the main plot in Fig. 4. Below the threshold value $P_t = 0.9$, only the symmetric (odd and even) solutions exist and their amplitudes almost superpose. At the threshold value, a new solution appears which is asymmetric with amplitudes A and B in the two arrays represented by the upper and lower branches in Fig. 4.

The existence of the asymmetric solution founds the existence of the self-trapping regime displayed in Fig. 2. The regime of Josephson oscillation is based on the coupled mode approach as follows. Using the symmetric and antisymmetric basic solutions, one builds a variational ansatz by seeking the solution $\psi(z,x)$ under the form

$$\begin{aligned} \psi(z,x) &= \psi_1(z)\Phi_1(x) + \psi_2(z)\Phi_2(x),\\ \sqrt{2}\Phi_1 = \Phi_+ + \Phi_-&, \qquad \sqrt{2}\Phi_2 = \Phi_+ - \Phi_-. \end{aligned} \tag{9}$$

The functions $|\psi_1(z)|^2$ and $|\psi_2(z)|^2$ are interpreted as the *probabilities* to find the light localized on the left or right arrays. By construction, the overlap of Φ_1 with Φ_2 is negligible; consequently, the projection of the NLS Eq. (4) successively on Φ_1 and Φ_2 provides the coupled mode equations [9,16]

$$i\frac{\partial\psi_1}{\partial z} + D|\psi_1|^2\psi_1 = r\psi_2, \qquad i\frac{\partial\psi_2}{\partial z} + D|\psi_2|^2\psi_2 = r\psi_1, \tag{10}$$

with coupling constant r and nonlinearity parameter D defined by

$$r = \frac{\int[(\partial_x\Phi_1)(\partial_x\Phi_2) + V\Phi_1\Phi_2]dx}{\int \Phi_1^2 dx}, \qquad D = \frac{\int \Phi_1^4 dx}{\int \Phi_1^2 dx}.$$

An explicit solution of (10) in terms of Jacobi elliptic functions has been found in Ref. [16] and used in Bose-Einstein condensates in Ref. [10]. It is a good approximation for the system in a double harmonic potential well [15] and correctly describes the oscillatory regime in our case. Indeed, when the power is initially injected into one array, say, $|\psi_1(0)| = 1$, $|\psi_2(0)| = 0$, we obtain for $D < 4r$

$$|\psi_1|^2 = \frac{1}{2}\left[1 + \mathrm{cn}\left(2rz, \frac{D}{4r}\right)\right], \qquad |\psi_2|^2 = 1 - |\psi_1|^2. \tag{11}$$

Since $|\psi_1|$ oscillates around the value 0, this expression describes an oscillation of light intensity between the left and the right array. The period of this oscillation is

$$T = 2\mathbb{K}(D/4r)/r \tag{12}$$

and has been checked on various numerical shots at different total input power. In summary, while the self-trapping regime is directly interpreted in terms of the asymmetric solution, the interpretation of the Josephson oscillation regime needs to call to the coupled mode approach, which, in turn, fails to explain the observed *coexistence* of both regimes.

Such a coexistence, however, is understood in terms of the energy (8) which can be evaluated, at given total power P_t, on the one side for the symmetric solution Φ_+ and on the other side for the asymmetric solution Φ_a. As shown in the inset in Fig. 4, these two energy values E_+ and E_a result to be very close up to the total power value $P_t \approx 2$. Consequently, switching from one regime to the other is allowed at fixed power. In particular, in the numerical experiments in Fig. 2, the total power and energy are the same before and after the local variation of the refractive index of the central waveguide.

It is worth noting that a similar analysis in the case of harmonic double well potential [14,15] shows that the energy of the asymmetric solution (if it exists) is significantly smaller than the energy of the symmetric solution. In such a situation, it thus is impossible to switch from a self-trapped state to an oscillatory regime when keeping the energy and total power constant.

Conclusion.—A new coherent state in weakly linked waveguide arrays has been discovered. This coherent state has the property of being bistable, and one can easily switch from oscillatory to self-trapping regimes and back. In the region of injected power where the asymmetric solution coexists with the symmetric and asymmetric stationary solutions, we have induced the switch from one regime to the other by varying the height of the barrier, that is, e.g., by introducing a defect in the central waveguide. In view of a real experiment, one could induce such flips by other methods, for instance, by applying external local perturbation such as a variation of the profile of the injected power.

We thank F. T. Arecchi, A. Montina, and S. Wimberger for useful discussions. R. K. acknowledges support by the Marie Curie Fellowship (No. MIF1-CT-2005-021328) and NATO Grant No. FEL.RIG.980767. We also acknowledge financial support under the PRIN05 grant on *Dynamics and thermodynamics of systems with long-range interactions*.

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