Quantum Open System Theory: Bipartite Aspects

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We demonstrate in straightforward calculations that even under ideally weak noise the relaxation of bipartite open quantum systems contains elements not previously encountered in quantum noise physics. While additivity of decay rates is known to be generic for decoherence of a single system, we demonstrate that it breaks down for bipartite coherence of even the simplest composite systems.

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In this Letter, we will establish results that contradict the long-standing belief that additivity of coherence decay rates is a natural consequence of weak noises. This belief means that the relaxation rate of any system exposed to a collection of weak noises is the sum of the relaxation rates associated with the noises separately. Although implied in many textbook discussions, an actual proof of additivity may be difficult to locate. We supply here a proof of additivity for a single qubit coupled to two independent weak noises (here amplitude noise and phase noise), but our main message is the demonstration of violations of additivity in the case of entanglement decay when two or more qubits are involved. That is, we will show that a quantum system with the most elementary composite structure (e.g., simply made of two distinct parts) need not and generally does not exhibit relaxation-rate additivity even though the separate parts do. This result is purely quantum mechanical and extends our understanding of the power of quantum coherence in an unexpected direction.

We now present an additivity proof that is quantum mechanical in order to eliminate from concern the possibility that quantum systems are intrinsically different from classical ones in their response to weak noise. We will consider ideal noise sources where "ideal" means that the noise is sufficiently weak and random that the noise-system interaction is both reliably linear and without significant backaction on the noise source. Each noise source can then be treated as a reservoir made of an infinite collection of random and very broadband harmonic oscillators at zero temperature.

Of course, the relaxing system need not be linear, so we choose the simplest nonlinear system, a qubit (two-level atom, spin one-half, etc.), for our example. The total Hamiltonian for a qubit coupled to two noise sources (two "environments") can be written as follows:

$$H_{\rm tot} = H_{\rm sys} + H_{\rm int} + H_{\rm env}, \tag{1}$$

where (with $\hbar = 1$)

$$H_{\rm sys} = \tfrac{1}{2}\omega_0\sigma_z \quad \text{and} \quad H_{\rm env} = \sum_\lambda \omega_\lambda a_\lambda^\dagger a_\lambda + \sum_\mu \nu_\mu b_\mu^\dagger b_\mu$$

are the Hamiltonians of the qubit system and two local environments. As an example of the types of relaxation that will be relevant, we suppose that the two environments couple in the one case longitudinally and in the other case transversely to the qubit. Thus, we have for the interaction of the qubit with its two different noise sources:

$$H_{\rm int} = k_1\sum_\lambda (f_\lambda^*\sigma_- a_\lambda^\dagger + f_\lambda\sigma_+ a_\lambda) + k_2\sum_\mu \sigma_z(g_\mu^* b_\mu^\dagger + g_\mu b_\mu). \tag{2}$$

We naturally assume that the two noise sources are not cross correlated, and in the ideal case under consideration they can be treated in the familiar Born-Markov limit. Thus, we write the longitudinal and transverse self-correlation functions in the form $\alpha_1(t,s) = \Gamma_1\delta(t-s)$ and $\alpha_2(t,s) = \Gamma_2\delta(t-s)$, respectively. The calculation of the time dependence of qubit coherence follows the usual rules [1,2], and we find, for longitudinal noise alone ($k_1 = 1$, $k_2 = 0$),

$$\rho_{12}(t) = e^{-i\omega_0 t}e^{-(1/2)\Gamma_1 t}\rho_{12}(0),$$

while for transverse noise alone ($k_1 = 0$, $k_2 = 1$), we have

$$\rho_{12}(t) = e^{-i\omega_0 t}e^{-\Gamma_2 t}\rho_{12}(0).$$

Now we switch on both longitudinal and transverse noise at the same time ($k_1 = k_2 = 1$). The master equation for the qubit system after tracing over two noise variables is simply given by (in the interaction picture)

$$\frac{d}{dt}\rho = \frac{\Gamma_1}{2}(2\sigma_-\rho\sigma_+ - \sigma_+\sigma_-\rho - \rho\sigma_+\sigma_-) + \frac{\Gamma_2}{2}(\sigma_z\rho\sigma_z - \rho). \tag{3}$$

The explicit solution of the above equation gives, for the qubit coherence,

$$\rho_{12}(t) = e^{-i\omega_0 t}e^{-((1/2)\Gamma_1+\Gamma_2)t}\rho_{12}(0). \tag{4}$$

This is all that is needed for a proof that the total internal decoherence rate of a qubit under ideal longitudinal and transverse noises applied at the same time is given by the sum of the separate rates: $\frac{1}{2}\Gamma_1 + \Gamma_2$. Finally, note that the linearity of the ideal noise interactions makes it obvious that any number of sources of longitudinal noise (any number of distinct $f_\lambda^{(n)} a_\lambda^{(n)} \sigma_+$ terms) will additively contribute to a total Γ_1, and, similarly, all $g_\mu^{(m)} b_\mu^{(m)} \sigma_z$ transverse noise sources will contribute to Γ_2.

To the present time, treatments of open quantum system theory [3–5] are based on this scenario in which a "small" system has a weak interaction with one or more reservoirs, and this is the cause of its relaxation (its loss of self-coherence). Now we extend the discussion very slightly and consider in detail the simplest quantum system made of two parts, a pair of qubits. Remarkably, this simple step takes us onto new ground within the theory of quantum open systems. We will show that the internal coherence of the two-qubit system exhibits a nonadditive response. We believe that this is the first demonstration of the effect.

In order to ensure focus on the main point, in the following we will not permit the two qubits to interact or communicate with each other and will allow them to be influenced only by noise sources that also have no contact with each other. The only connection between the parts of the two-qubit system will be pure information. Thus, the Hamitonian for the two-qubit case is simply the addition of the Hamiltonians (1) for the two qubits, respectively. Two-party aspects of quantum information such as mixed states and entanglement are not present in any single system or in any pair of classical systems and can lead to new open-system effects. Although more general results can be obtained using our methods, we will concentrate on mutual entanglement as the most useful measure of bipartite coherence for our demonstration. To determine entanglement quantitatively, we will use concurrence [6].

Solutions of the appropriate (Born-Markov) equations for noisy evolution of two-qubit density matrices can be obtained via several routes [7], and we find the Kraus operator form [8] most convenient. Given a state ρ (pure or mixed), its evolution can be written compactly as

$$\rho(t) = \sum_\mu K_\mu(t)\rho(0)K_\mu^\dagger(t), \tag{5}$$

where the so-called Kraus operators K_μ satisfy $\sum_\mu K_\mu^\dagger K_\mu = 1$ for all t.

In order to demonstrate the breakdown of additivity for ideally weak noises, it suffices to find a two-party state that experiences continuous exponential decay under each noise but fails to do so when two or more noises are active at the same time. Actually, we can identify an entire class of such states. What is more, the class is widely known to be relevant in a variety of physical situations including pure Bell states [9] and the Werner mixed state [10] as special cases.

This class of bipartite states is represented by the following two-qubit density matrix, where we use conventional ordering of rows and columns related to eigenstates of σ_z^A and σ_z^B in the sequence $[++, +-, -+, --]$:

$$\rho^{AB} = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & z & 0 \\ 0 & z^* & c & 0 \\ 0 & 0 & 0 & d \end{pmatrix}. \tag{6}$$

Obviously, $a + b + c + d = 1$. We easily find the concurrence of this state to be given by

$$C^{AB} = 2\max\{0, |z| - \sqrt{ad}\}. \tag{7}$$

For even greater simplicity, within this set of density matrices we will first focus on a smaller subcategory with a single positive parameter λ:

$$\rho_\lambda^{AB} = \frac{1}{9}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & \lambda & 0 \\ 0 & \lambda & 4 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{8}$$

Initially, for ρ_λ^{AB} we have $C_\lambda(0) = 2\lambda/9$.

To begin our time-dependent calculations, we consider pure transverse (phase) noise, for which we have the following compact Kraus operators for independently evolving qubits A and B:

$$K_1 = \begin{pmatrix} \gamma_A & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} \gamma_B & 0 \\ 0 & 1 \end{pmatrix}, \tag{9}$$

$$K_2 = \begin{pmatrix} \gamma_A & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 0 & \omega_B \end{pmatrix}, \tag{10}$$

$$K_3 = \begin{pmatrix} 0 & 0 \\ 0 & \omega_A \end{pmatrix} \otimes \begin{pmatrix} \gamma_B & 0 \\ 0 & 1 \end{pmatrix}, \tag{11}$$

$$K_4 = \begin{pmatrix} 0 & 0 \\ 0 & \omega_A \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 0 & \omega_B \end{pmatrix}, \tag{12}$$

where the time-dependent Kraus matrix elements are

$$\gamma_A(t) = \exp[-\Gamma_2^A t/2] \quad \text{and} \quad \omega_A(t) = \sqrt{1 - \gamma_A^2(t)},$$

and similar expressions for $\gamma_B(t)$ and $\omega_B(t)$. We take $\Gamma_2^A = \Gamma_2^B \equiv \Gamma_2$ for greatest simplicity. We note that both of the mixed states written above have the property that they retain their form under these Kraus operators. For pure dephasing noise, the diagonal elements are constant $[a(t) = 1/9,\ d(t) = 0]$ and the Kraus operators give $z = \frac{\lambda}{9} \rightarrow z(t) = \frac{\lambda}{9}\exp[-\Gamma_2 t]$, and then the phase-noise concurrence decays asymptotically exponentially:

$$C_\lambda^{\text{ph.}}(t) = (2\lambda/9)\exp[-\Gamma_2 t]. \tag{13}$$

For longitudinal (amplitude) noise, the Kraus operators are slightly different (see [11]), but a direct calculation for the one-parameter example above gives

$$z = \frac{\lambda}{9} \rightarrow z(t) = \frac{\lambda}{9} \exp[-\Gamma_1 t], \tag{14}$$

$$a = \frac{1}{9} \rightarrow a(t) = \frac{1}{9} \exp[-2\Gamma_1 t], \tag{15}$$

$$d = 0 \rightarrow d(t) = \frac{1}{9}\omega_1^4 + \frac{8}{9}\omega_1^2, \tag{16}$$

where $\omega_1 = \sqrt{1 - \exp[-\Gamma_1 t]}$ and Γ_1 is the longitudinal decay rate for amplitude noise. From these, one easily shows that in the range $4 \geq \lambda \geq 3$ the amplitude-noise concurrence is given by

$$C_\lambda^{\text{am.}}(t) = \tfrac{2}{9}[\lambda - \sqrt{\omega_1^4 + 8\omega_1^2}]\exp[-\Gamma_1 t]. \tag{17}$$

Therefore, our bipartite entanglement under amplitude noise also decays smoothly and asymptotically exponentially to zero. With these two exercises in hand, we conclude that for our mixed two-party system the entanglement decays asymptotically smoothly to zero in the presence of either weak amplitude noise or weak phase noise.

Now we consider the issue of additivity and allow the weak noises to be applied together. All two-party density matrix elements decay at the sum of their respective phase and amplitude rates, as the additivity theorem ensures. However, for the entanglement measure of coherence, the consequence is strikingly different. Because the off-diagonal element $z = \lambda/9$ decays at the sum of the separate phase- and amplitude-noise rates, the two-noise concurrence takes the form:

$$C_\lambda^{\text{ph.+am.}}(t) = 2\max\{0, \lambda e^{-\Gamma_2 t} - \sqrt{\omega_1^4 + 8\omega_1^2}\}, \tag{18}$$

which has lost any trace of relaxation additivity, particularly the property of an asymptotically smooth approach of entanglement to zero. As we show in Fig. 1, over a continuous range of physical λ values, $C_\lambda^{\text{ph.+am.}}(t)$ actually goes abruptly to zero in a finite time and remains zero thereafter. This is the effect that has been called "entanglement sudden death" (ESD) [12], and it arises here more

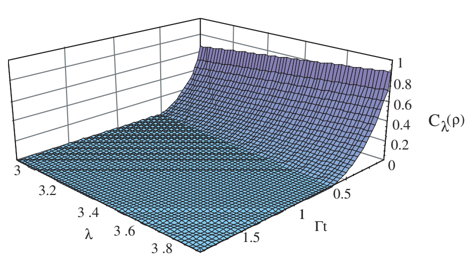

FIG. 1 (color online). The graph shows C_λ vs λ and t under the influence of combined phase and amplitude noise. The consequence is that (8) disentangles completely and abruptly in just a finite time for all λ in the range shown.

or less from nowhere, since there is no local effect, under the action of either of the weak noises, indicating that it should be expected. The present result shows that ESD is one consequence not previously noted, indicating necessary departures from standard elements of open-system theory in multiparty relaxation, even for ideally weak noise influences.

It is important to emphasize that our special one-parameter example is not a singular case. The simplest verifications of this can be made by just retaining $d = 0$ within the more general matrix class (6). The outcome of fairly straightforward calculations for the entire class is illustrated by the diagrams in Fig. 2. Part (i) shows that under pure amplitude noise either ESD or pure exponential decay may occur, with the boundary between them given just by $a = |z|^2$. Part (ii) indicates that for the entire range of a and $|z|$ under pure phase noise the decay is purely exponential. However, (iii) shows that under the combination of phase and amplitude noise every initial state (6) will disentangle abruptly. This directly shows that, when the parameters lie in the zone $a \leq |z|^2$, nonadditivity occurs for entanglement decay rates.

We end our examination with a general observation. The calculations displayed here reach only a small corner of a new domain of noise physics. Wider questions can also be answered. What if one applies both phase noise and amplitude noise to subsystem A alone, leaving B totally noise-free? One finds that this is enough to impact the bipartite AB entanglement just as strongly as before. What if one applies only phase noise to subsystem A and only amplitude noise to B? In that case, both A and B have to relax normally, but their mutual entanglement does not. These results can be verified by straightforward calculations. The fact that the same conclusion applies no matter where one looks in this domain demonstrates that information about an open bipartite quantum system will become degraded

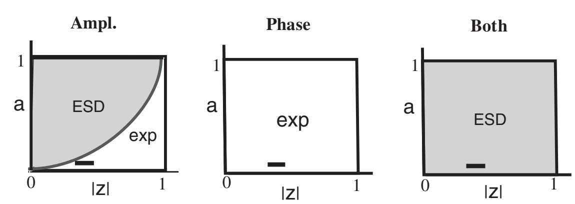

FIG. 2. This diagram shows the dramatic effect of the combination of two noises, amplitude and phase noise in this case, on all initial states (6) with $d = 0$. (i) Amplitude noise can lead to entanglement sudden death (dark zone), but for a large parameter range (white zone, $a \leq |z|^2$) the entanglement only decays exponentially. (ii) Under phase noise, the initial entanglement always decays exponentially (white zone). (iii) However, when the noises are combined, *all initally entangled states* suffer sudden death (dark zone). In each part, the solid-line segment ($a = 1/9$, $1/3 \leq |z| \leq 4/9$) shows the parameter range associated with the particularly simple concurrence C_λ that we discussed in detail.

with time as an indivisible quantum unit, no matter how its parts are engaged by weak noises, and the degradation is not predicted by the familiar smoothly decaying behavior familiar from the quantum theory of single open systems.

To summarize, in this Letter we introduced a commonly occurring category of two-system mixed states, shown in (6). By following their time-dependent behavior under the influence of ideally weak noises, we demonstrated the presence of elements of open quantum system theory not previously encountered. These become interesting whenever a small system has different quantum parts that can be entangled. Exactly this situation will arise, for example, in a quantum computer, where it is most desirable that two qubits retain a nonzero degree of mutual cross entanglement. It must be emphasized that none of our key AB results come from interaction or communication between the A and B parts of the two-party system or between their separate reservoirs.

This is perhaps the most striking aspect of the properties described: They are properties of joint-system information rather than joint-system interaction. To the extent that joint-system information is a resource of substantial value in one or another practical application of qubit networks, this aspect of time-dependent entanglement will be important. At the same time, it illuminates further the difficult fundamental challenge to understand the nature of coherence in multipartite mixed states, particularly in its time-dependent behavior, which has recently come under examination in both continuous spaces [13–15] and discrete spaces (qubit pairs [16–20], finite spin chains, and elementary lattices [21–24]) and decoherence dynamics in adiabatic entanglement [25], as well as in situations without relaxation [26] and in connection with direct entanglement observation [27]. These have all contributed to increased awareness of this domain.

Finally, it should be emphasized that, although entanglement measured by concurrence is not an observable represented by an Hermitian operator, nevertheless it is still possible to express the concurrence (7) in terms of the expectation values of certain ordinary physical observables [28]. Moreover, the recent proposals to directly measure the dissipative entanglement evolution have opened up a possibility of experimentally demonstrating the onset of the nonadditivity when nonlocal coherence decay is concerned [27,29].

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