Codimension-Two Critical Behavior in Vacuum Gravitational Collapse

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We consider the critical behavior at the threshold of black-hole formation for the five-dimensional vacuum Einstein equations satisfying the cohomogeneity-two triaxial Bianchi type-IX ansatz. Exploiting a discrete symmetry present in this model we predict the existence of a codimension-two attractor. This prediction is confirmed numerically and the codimension-two attractor is identified as a discretely self-similar solution with two unstable modes.

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Introduction.—Since the pioneering work of Choptuik on the collapse of a self-gravitating scalar field [1], the nature of the boundary between dispersion and black-hole formation in gravitational collapse has been a very active research area (see [2] for a review). One of the most intriguing aspects of these studies is the occurrence of discretely self-similar critical solutions. Discrete self-similarity (DSS) means invariance under rescalings of space and time variables by a constant factor e^{Δ}, where Δ is a number, usually called the echoing period. Critical solutions possessing this curious symmetry (with different echoing periods) have been found for several self-gravitating matter models (massless scalar field [1], Yang-Mills field [3], the σ model [4], and a few others) and recently also in vacuum gravitational collapse in higher dimensions [5,6].

Our current understanding of DSS solutions is very limited in comparison with continuously self-similar (CSS) solutions. In the case of spherical symmetry the CSS ansatz leads to a system of ordinary differential equations which can be handled analytically, and sometimes even rigorous proofs of its existence are feasible. For example, the existence of a countable family of CSS solutions was proved in the Einstein-sigma model [7] and the ground state of this family was identified as the critical solution (which was known previously from numerical studies of critical collapse). In contrast, the DSS ansatz leads to a $1+1$-dimensional eigenvalue problem which seems intractable analytically. Although this eigenvalue problem can be solved numerically, as was done by Gundlach for two models (scalar field [8] and Yang-Mills [9]), the numerical iterative procedure requires a good initial seed in order to converge. Thus, Gundlach's method is efficient in validating and refining DSS solutions which are already known from direct numerical simulations, but it is not useful in searching for new solutions.

In this Letter we consider the critical collapse for the five-dimensional vacuum Einstein equations satisfying the cohomogeneity-two triaxial Bianchi type-IX ansatz and provide heuristic arguments and numerical evidence for the existence of a DSS solution with two unstable modes. This is a continuation of our studies in [5] where we have shown the existence of a critical DSS solution with one unstable mode and the associated type-II critical behavior in this model. On the basis of our result it is tempting to conjecture that the critical DSS solution is a ground state of a countable family of DSS solutions with an increasing number of unstable modes.

Background.—Our starting point is the cohomogeneity-two symmetry reduction of the Einstein equations in five dimensions, which is based on the following ansatz introduced by us in [5]:

$$ds^2 = -Ae^{-2\delta}dt^2 + A^{-1}dr^2 + \tfrac{1}{4}r^2[e^{2B}\sigma_1^2 + e^{2C}\sigma_2^2 + e^{-2(B+C)}\sigma_3^2], \tag{1}$$

where A, δ, B, and C are functions of time t and radius r. The angular part of (1) is the $SU(2)$-invariant homogeneous metric on the squashed three-sphere with σ_k being standard left-invariant one-forms on $SU(2)$:

$$\sigma_1 + i\sigma_2 = e^{i\psi}(\cos\theta d\phi + id\theta), \qquad \sigma_3 = d\psi - \sin\theta d\phi, \tag{2}$$

where $0 \le \theta \le \pi$, $0 \le \phi \le 2\pi$, $0 \le \psi \le 4\pi$ are the Euler angles. The squashing modes, B and C, play the role of dynamical degrees of freedom. This ansatz provides a simple $1+1$-dimensional framework for investigating the dynamics of gravitational collapse in vacuum. In [5] we made a simplifying assumption that $B = C$, which means that the ansatz (1) has an additional $U(1)$ symmetry and only one dynamical degree of freedom (the so-called biaxial case). In this Letter we drop this assumption and consider the full triaxial case with two dynamical degrees of freedom.

Substituting the ansatz (1) into the vacuum Einstein equations in five dimensions and using the mass function $m(t, r)$, defined by $A = 1 - m/r^2$, we get the following system:

$$m' = 2r^3[e^{2\delta}A^{-1}(\dot{B}^2 + \dot{C}^2 + \dot{B}\dot{C}) + A(B'^2 + C'^2 + B'C')] + \tfrac{2}{3}r(3 + e^{4B} + e^{4C} - 2e^{-2B} - 2e^{-2C} - 2e^{2(B+C)} + e^{-4(B+C)}), \quad (3a)$$

$$\dot{m} = \tfrac{2}{3}r^3A(\dot{C}B' + \dot{B}C' + 2\dot{B}B' + 2\dot{C}C'), \quad (3b)$$

$$\delta' = -\tfrac{2}{3}r[e^{2\delta}A^{-2}(\dot{B}^2 + \dot{C}^2 + \dot{B}\dot{C}) + B'^2 + C'^2 + B'C'], \quad (3c)$$

$$(e^{\delta}A^{-1}r^3\dot{B})^{\cdot} = (e^{-\delta}Ar^3B')' + \tfrac{4}{3}e^{-\delta}r(2e^{4B} + 2e^{-2B} - e^{2(B+C)} - e^{-4(B+C)} - e^{4C} - e^{-2C}), \quad (3d)$$

$$(e^{\delta}A^{-1}r^3\dot{C})^{\cdot} = (e^{-\delta}Ar^3C')' + \tfrac{4}{3}e^{-\delta}r(2e^{4C} + 2e^{-2C} - e^{2(B+C)} - e^{-4(B+C)} - e^{4B} - e^{-2B}), \quad (3e)$$

where primes and overdots denote derivatives with respect to r and t, respectively.

If $B = C = 0$, then the ansatz (1) reduces to the standard spherically symmetric metric for which the Birkhoff theorem applies and the only solutions are Minkowski ($\delta = 0$, $m = 0$) and Schwarzschild ($\delta = 0$, $m = \text{const} > 0$). We showed in the biaxial case [5] that these two static solutions play the role of attractors in the evolution of generic regular initial data (small and large ones, respectively). We have verified that this property remains true in the triaxial case (see also [10]). Note that equations for small perturbations δB and δC around the Minkowski and Schwarzschild solutions decouple, hence linear stability results and the decay rates discussed in [5] carry over without any changes.

Heuristics.—Below we focus on initial data on a borderline between dispersion and collapse. Our preliminary studies of the evolution of such data suggested that the relaxation of symmetry from biaxial to triaxial does not change the phenomenology of critical behavior observed in [5]; that is, we have found our old biaxial DSS solution (hereafter referred to as the DSS_1) as the critical solution. In other words the biaxial symmetry seems to be recovered dynamically not only for generic initial data but also for the critical ones (see Fig. 1). The mechanism of this nonlinear synchronization phenomenon is an open problem which we hope to pursue elsewhere. Here we take it as a fact and use it as the starting point in further discussion.

The second key element of our argument is the fact that the system (3) possesses a discrete Z_3 symmetry which corresponds to the freedom of permutations of coefficients of one-forms σ_k in the angular part of the metric (1). These permutations are generated by the following transpositions:

$$T_{12}\colon (B, C) \to (C, B), \qquad T_{23}\colon (B, C) \to (B, -B - C),$$
$$T_{13}\colon (B, C) \to (-B - C, C), \quad (4)$$

where the transposition T_{ij} swaps the coefficients of σ_i^2 and σ_j^2 in (1). Biaxial configurations correspond to the fixed points of these transpositions: (B, B), $(B, -B/2)$, and $(B, -2B)$.

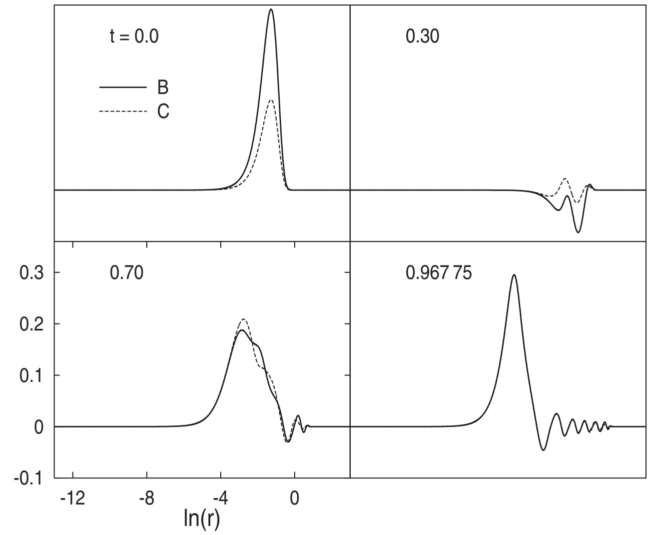

FIG. 1. Snapshots of the evolution of near-critical initial data of the form (5) for $a = 1/2$. In the last frame the squashing modes become synchronized and the solution coincides with the DSS_1.

Thus, each biaxial solution, in particular, the DSS_1, exists in three different but geometrically equivalent copies. Let $\mathcal{M}_{\mathrm{crit}}$ denote the codimension-one critical surface in the phase space which separates dispersion from collapse and let $\mathcal{M}_i$ ($i = 1, 2, 3$) denote the basins of attraction of three symmetry-related copies of the DSS_1 solution. Since $\mathcal{M}_i$ lie in $\mathcal{M}_{\mathrm{crit}}$ and $\mathcal{M}_{\mathrm{crit}}$ is connected, there should exist codimension-two boundaries that separate different $\mathcal{M}_i$ from each other. It is natural to expect that these boundaries are given by the stable manifolds of three symmetry-related copies of a solution with two unstable modes. In the next section, we confirm this picture numerically and demonstrate that the codimension-two attractor is discretely self-similar.

Numerics.—In order to find a codimension-two attractor predicted above, we consider the two-parameter family of time-symmetric initial data of the form

$$B(0, r) = pf(r), \qquad C(0, r) = aB(0, r), \quad (5)$$

where $f(r)$ is a smooth localized function satisfying regularity conditions at $r = 0$. The results presented below were produced for the generalized Gaussian $f(r) = 100r^2\exp[-20(r - 0.1)^2]$. We pick some value of the parameter a and then, using bisection, we fine-tune the parameter p to the critical value $p^*(a)$. We are interested in how the phenomenology of critical behavior depends on a. There are three distinguished values of a which correspond to three biaxial configurations: $a = 1$, $a = -1/2$, or $a = -2$. Since biaxiality is preserved during evolution, for these special initial data we obviously find our old DSS_1 as the critical solution [note that the corresponding critical values of p are related by the symmetry (4) $p^*(-1/2) = -2p^*(1)$ and $p^*(-2) = p^*(1)$, which provides a useful test of the accuracy of numerics]. As we have already mentioned, numerical studies of critical collapse indicate that for other (generic) values of the parameter a the DSS_1 solution, modulo the Z_3 symmetry, also acts as an attractor. Let us denote by $X_1^{(1)}$, $X_2^{(1)}$, and $X_3^{(1)}$ the three copies of the DSS_1 solution, corresponding to $a = 1$, $a = -1/2$, and $a = -2$, respectively. Below we focus our attention on $X_1^{(1)}$ and $X_2^{(1)}$. By continuity, the solution $X_1^{(1)}$ is the attractor for the values of a close to 1 and the solution $X_2^{(1)}$ is the attractor for the values of a close to $-1/2$. Thus, in the interval $-1/2 < a < 1$ there must exist at least one critical value a^* such that the critical initial data (5) corresponding to $a^* \pm \epsilon$ (for a sufficiently small ϵ) evolve to the different copies $X_1^{(1)}$ and $X_2^{(1)}$. As usual, this critical value a^* can be determined by bisection [11]. We shall refer to initial data with parameters $[a^*, p^*(a^*)]$ as double critical. In Fig. 2 we show the evolution of near double critical initial data. The key new feature which is apparent in Fig. 2 (in contrast to Fig. 1) is the occurrence of another intermediate attractor around which the solution hangs for a while before approaching the DSS_1 attractor. We find that the new attractor is discretely self-similar with the echoing period $\Delta_2 \approx 0.395$ (see Fig. 3). We call this solution the DSS_2 and denote it by $X^{(2)}$.

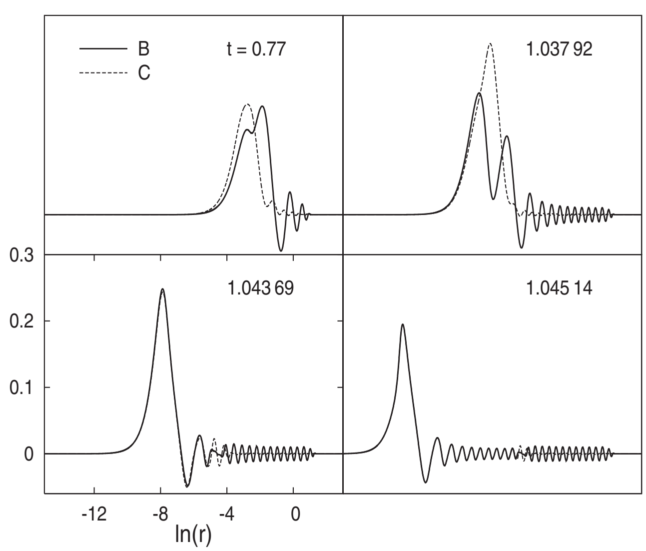

FIG. 2. Snapshots of the evolution of near double critical initial data ($a = 0.1411\ldots$, $p = 0.0952\ldots$). The new (codimension-two) intermediate attractor (DSS_2) is seen in the second frame. The departure from this attractor (proceeding from the origin) and the approach to the old (codimension-one) intermediate attractor ($X_1^{(1)}$ copy of the DSS_1) in seen in the third frame. In the last frame the DSS_1 regime is well developed near the origin while the DSS_2 regime is still visible for large r. To produce this figure we had to fine-tune p to the critical value with a relative accuracy of 10^{-26} (using quadruple precision arithmetics).

The behavior seen if Fig. 2 has a natural explanation within the heuristic picture sketched above. According to this picture the DSS_2 has two unstable modes, one tangential and one transversal to the critical surface. During the DSS_2 intermediate regime the departure from $X^{(2)}$ is well approximated by a linear combination of two unstable modes:

$$\delta X^{(2)} \approx c_1(a, p)e^{\lambda_1^{(2)}\tau} f_1(\rho, \tau) + c_2(a, p)e^{\lambda_2^{(2)}\tau} f_2(\rho, \tau), \tag{6}$$

where $\rho = r/(T - t)$ and $\tau = -\ln(T - t)$ are the similarity variables (T is the blowup time of the DSS_2), $f_i(\rho, \tau)$ are the eigenfunctions (periodic in τ with the period Δ_2), and $\lambda_i^{(2)} > 0$ are the respective eigenvalues. The coefficients c_i are the only vestige of initial data. Given a close to a^*, for the perfect fine-tuning of p to its critical value, the transversal unstable mode (let it be f_1) is completely

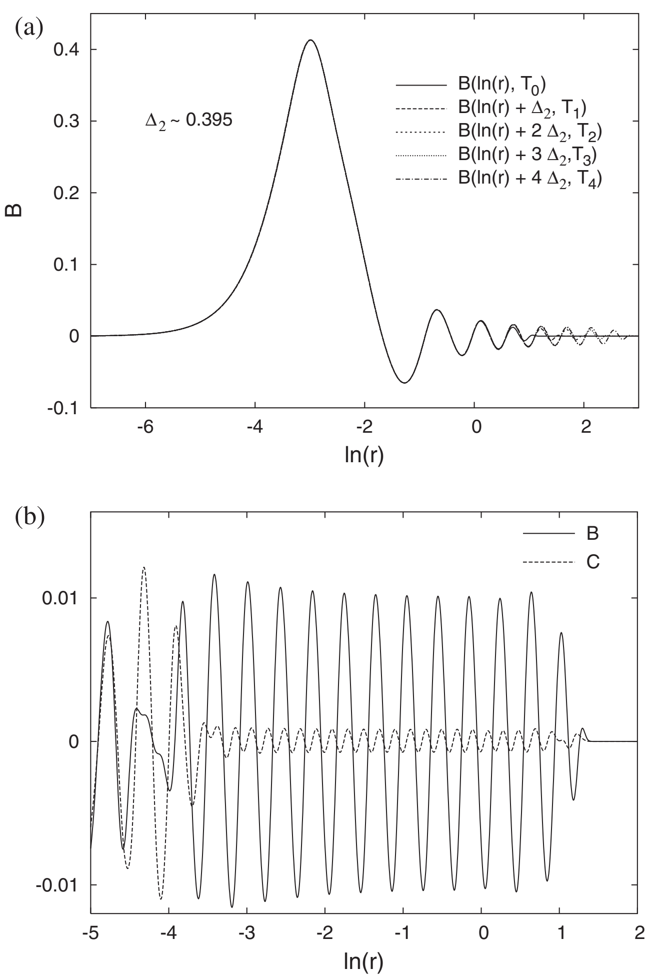

FIG. 3. (a) Evidence for discrete self-similarity of the codimension-two attractor. We plot the mode B at some central proper time T_0 during the intermediate regime shown in the second frame in Fig. 2 and superimpose the next four echoes which subsequently develop. (b) The close-up of the DSS_2 solution. It is evident that this solution is not biaxial. Note that the mode C oscillates with twice the frequency of the mode B.

suppressed, that is, $c_1(a, p^*(a)) = 0$. In practice, we can fine-tune p with very high precision so that $c_1(a, p(a))$ is almost zero. For such data, $c_1 \ll c_2$, hence after several cycles the solution leaves the DSS_2 along the tangential unstable direction and then, after a short transient period, approaches one of the copies $X_1^{(1)}$ or $X_2^{(1)}$ (depending on the sign of c_2) of the DSS_1. Afterwards the solution stays near DSS_1 for some time and eventually moves away from the critical surface $\mathcal{M}_{\rm crit}$ towards collapse or dispersion.

The existence of the DSS_2 solution has an interesting consequence for the black-hole mass scaling law, $M_{\rm BH} \sim (p - p^*)^\gamma$, associated with the type-II critical collapse. Consider a near-critical value a and monitor the black-hole mass over a wide range of supercritical values $p > p^*(a)$. As we have discussed above, for p very close to the threshold the solution moves away from the critical surface along the unstable mode of the DSS_1. However, as $p - p^*$ increases, we observe a competition between the transversal and the tangential modes of the DSS_2. Far from the threshold the transversal mode becomes dominant ($c_1 \gg c_2$), which means that the solution moves away from the critical surface along the transversal unstable direction of the DSS_2 and never approaches the DSS_1. According to the well-known argument from dimensional analysis [2], the scaling exponent γ is related to the eigenvalue λ of the unstable mode along which the solution is ejected towards collapse: $\gamma = 2/\lambda$ (the factor of 2 comes from the fact that in five dimensions mass has dimension length^2). Thus, for near double critical initial data the mass scaling law is expected to exhibit the crossover from the exponent $\gamma_1 = 2/\lambda^{(1)}$ (where $\lambda^{(1)}$ is the eigenvalue of the single unstable mode of the DSS_1) for small $p - p^*$ to $\gamma_2 = 2/\lambda_1^{(2)}$ for relatively large $p - p^*$ (see Fig. 4 for the numerical confirmation of this prediction). Using this relationship and the numerically determined γ_2 we estimate that $\lambda_1^{(2)} \approx 6.67$. The second eigenvalue $\lambda_2^{(2)}$ associated with the tangential unstable mode of the DSS_2 can be obtained from the relationship $\lambda_2^{(2)}\Delta_2\delta n = -\delta\pi$, where δn and $\delta\pi$ are the changes in the echo number n and $\pi = \ln|p - p^*|$. Fitting this formula to numerical data we obtain $\lambda_2^{(2)} \approx 6.03$.

We remark that the heuristic argument for the existence of a codimension-two attractor presented in this Letter can be applied repeatedly to argue for the existence of higher codimension attractors [12]; however, the numerical search for these solutions via multiparameter fine-tuning would be extremely difficult.

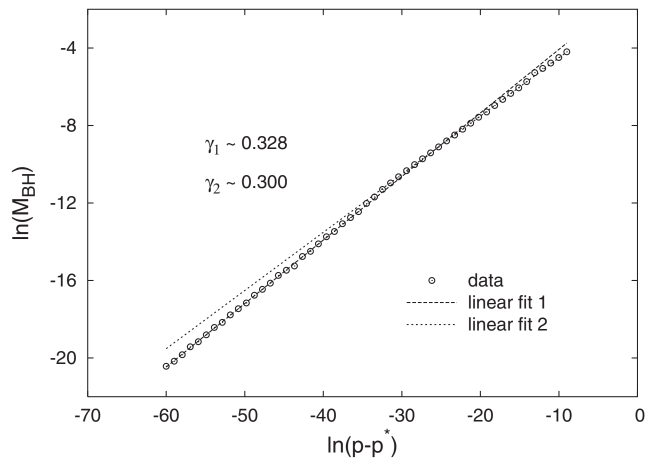

FIG. 4. The black-hole mass scaling law for near double critical initial data. The linear fit yields different slopes for small and large distances from the threshold.

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[11] Actually, it turns out that there are many (perhaps infinitely many) critical values of a, intermingled in a complicated way. This suggests that the structure of codimension-two boundaries between different basins of attractions $\mathcal{M}_i$ is very complex, and quite likely fractal.
[12] Cf. K. Corlette and R. M. Wald, Commun. Math. Phys. **215**, 591 (2001), where a similar in spirit argument is used to prove the existence of infinitely many harmonic maps between n-spheres.