Generic Entanglement and Standard Form for *N*-Mode Pure Gaussian States

Gerardo Adesso

Dipartimento di Fisica "E. R. Caianiello", Università degli Studi di Salerno, INFN Sezione di Napoli-Gruppo Collegato di Salerno,
Via S. Allende, 84081 Baronissi (SA), Italy
Centre for Quantum Computation, DAMTP, Centre for Mathematical Sciences, University of Cambridge, Wilberforce Road,
Cambridge CB3 0WA, United Kingdom

We investigate the correlation structure of pure *N*-mode Gaussian resources which can be experimentally generated by means of squeezers and beam splitters, whose entanglement properties are *generic*. We show that those states are specified (up to local unitaries) by $N(N - 1)/2$ parameters, corresponding to the two-point correlations between any pair of modes. Our construction yields a practical scheme to engineer such generic-entangled *N*-mode pure Gaussian states by linear optics. We discuss our findings in the framework of Gaussian matrix product states of harmonic lattices, raising connections with entanglement frustration and the entropic area law.

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Introduction.—Multipartite entanglement in pure states of many systems is a founding property and a crucial resource for quantum information science, yet its complete theoretical understanding is still lacking. A basic property of entanglement is its invariance under unitary operations performed locally on the subsystems. To describe entanglement efficiently is thus natural to lighten quantum systems of the unnecessary degrees of freedom adjustable by local unitaries (LUs), and to classify states according to *standard forms* representative of LU equivalence classes [1]. Alongside the traditional qubit-based approach, quantum information with continuous variables (CV) is a burgeoning field mainly spinning around the theory and applications of entanglement in Gaussian states [2].

In this Letter we address the question of how many physical resources are really needed to engineer and characterize entanglement in pure Gaussian states of an arbitrary number of modes, up to LU operations. For states of $N \leq 3$ modes, it has been shown that such a number of minimal degrees of freedom scales as $N(N - 1)/2$ [3,4]. For a higher number of modes, however, a richer structure is achievable by pure Gaussian states, as from the normal form of Ref. [5] a minimal number of parameters given by $N(N - 2)$ can be inferred [6]. A random state of $N \geq 4$ modes, selected according to the uniform distribution over pure Gaussian states, will be thus reducible to a form characterized by such a number of independent quantities. However, in practical realizations of CV quantum information one is interested in states which, once prepared with efficient resources, still achieve an almost complete structural variety in their multipartite entanglement properties. Such states will be said to possess *generic entanglement* [7], where generic means practically equivalent to that of random states, but engineered (and described) with a considerably smaller number of degrees of freedom.

Precisely, we define as "generic-entangled" those Gaussian states whose local entropies of entanglement in any single mode are independent, and bipartite entanglements between any pair of modes are unconstrained. Having a standard form for such *N*-mode Gaussian states, may be in fact extremely helpful in understanding and quantifying multipartite CV entanglement, in particular, from the theoretical point of view of entanglement sharing and monogamy constraints [4,8], and from a more pragmatical approach centered on using entanglement as a resource. We show that, to achieve generic entanglement, for the global pure *N*-mode Gaussian state it is enough to be described by a minimal number of parameters (corresponding to the LU-invariant degrees of freedom) equal to $N(N - 1)/2$ for any *N*, and thus much smaller than the $2N(2N + 1)/2$ of a completely general covariance matrix. Therefore, generic entanglement appears in states which are highly *not* "generic" in the sense usually attributed to the term, i.e., randomly picked. Crucially, we demonstrate that generic-entangled Gaussian states coincide with the resources typically employed in experimental realizations of CV quantum information [2], and we provide an optimal scheme for their state engineering.

Preliminaries.—We consider a CV system consisting of *N* canonical bosonic modes, and described by the vector $\hat{X} = \{\hat{q}_1, \hat{p}_1, \hat{q}_2, \hat{p}_2, \ldots, \hat{q}_N, \hat{p}_N\}$ of the field quadrature operators, which satisfy the commutation relations $[\hat{X}_i, \hat{X}_j] = 2i\Omega_{ij}$, with the symplectic form $\boldsymbol{\Omega} = \omega^{\oplus N}$ and $\omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Gaussian states (such as vacua, coherent, and squeezed states) are defined by having a Gaussian characteristic function in phase space [2]. They are fully characterized by the first statistical moments (arbitrarily adjustable by LUs: we will set them to zero) and by the $2N \times 2N$ covariance matrix (CM) σ of the second moments $\sigma_{ij} = \langle\{\hat{X}_i, \hat{X}_j\}\rangle/2 - \langle\hat{X}_i\rangle\langle\hat{X}_j\rangle$. The CM σ of an

arbitrary N-mode Gaussian state can be written as follows in terms of 2×2 submatrices

$$\sigma = \begin{pmatrix} \sigma_1 & \cdots & \varepsilon_{1N} \\ \vdots & \ddots & \vdots \\ \varepsilon_{1N}^T & \cdots & \sigma_N \end{pmatrix}. \qquad (1)$$

Symplectic operations (i.e., belonging to the group $Sp_{(2N,\mathbb{R})} = \{S \in SL(2N, \mathbb{R}): S^T \boldsymbol{\Omega} S = \boldsymbol{\Omega}\}$) acting by congruence on CMs in phase space, amount to unitary operations on density matrices in Hilbert space. Any N-mode Gaussian state can be transformed by symplectic operations in its Williamson diagonal form [9] ν, such that $\sigma = S^T \nu S$, with $\nu = \text{diag}\{\nu_1, \nu_1, \ldots \nu_N, \nu_N\}$. The quantities $\nu_i \geq 1$ are the symplectic eigenvalues of σ [10].

We define the *symplectic rank* $\aleph$ of a CM as the number of its symplectic eigenvalues different from 1, corresponding to the number of nonvacua normal modes. Any *pure* state has $\aleph = 0$. The CM σ^p of any N-mode pure Gaussian state satisfies the matrix identity [5] $-\boldsymbol{\Omega} \sigma^p \boldsymbol{\Omega} \sigma^p = 1$. As a consequence of the Schmidt decomposition, applied at the CM level [11] for the bipartition $i|(1, \ldots, i-1, i+1, \ldots, N)$, any $(N-1)$-mode reduced CM of the CM σ^p of a N-mode pure Gaussian state has symplectic rank $\aleph = 1$.

Minimal number of parameters.—Adopting the above definition of generic entanglement, we prove now the main *Proposition 1*: a generic-entangled N-mode pure Gaussian state is described, up to local symplectic (unitary) operations, by $N(N-1)/2$ independent parameters.

Proof.—Let us start with a N-mode pure state, described by a CM $\sigma^p \equiv \sigma$ with all single-mode blocks in diagonal form: we can always achieve this by local single-mode Williamson diagonalizations in each of the N modes. Let $\sigma^{\backslash 1} \equiv \sigma_{2,\ldots,N}$ be the reduced CM of modes $(2, \ldots, N)$. It can be diagonalized by means of a symplectic $S_{2,\ldots,N}$, and brought thus to its Williamson normal form, characterized by a symplectic spectrum $\{a, 1, \ldots, 1\}$, where $a = \sqrt{\text{Det}\sigma_1}$. Transforming σ by $S = \mathbb{1}_1 \oplus S_{2,\ldots,N}$, brings the CM into its Schmidt form, constituted by a two-mode squeezed state between modes 1 and 2 (with squeezing a), plus $N - 2$ vacua [11,12].

All N-mode pure Gaussian states are thus completely specified by the symplectic $S_{2,\ldots,N}$, plus the parameter a. Alternatively, the number of parameters of σ is also equal to those characterizing an arbitrary mixed $N - 1$ Gaussian CM, with symplectic rank $\aleph = 1$ (i.e., with $N - 2$ symplectic eigenvalues equal to 1). This means that, assigning the reduced state $\sigma_{2,\ldots,N}$, we have provided a complete description of σ. In fact, the parameter a is determined as the square root of the determinant of the CM $\sigma_{2,\ldots,N}$.

We are now left to compute the minimal set of parameters of an arbitrary mixed state of $N - 1$ modes, with symplectic rank $\aleph = 1$. While we know that for $N \geq 4$ this number is equal to $N(N - 2)$ in general [5], we want to prove that for generic-entangled Gaussian resource states

this number reduces to

$$\Xi_N = N(N-1)/2. \qquad (2)$$

We prove it by induction. For a pure state of one mode only, there are no reduced "zero-mode" states, so the number is zero. For a pure state of two modes, an arbitrary one-mode mixed CM with $\aleph = 1$ is completely determined by its own determinant, so the number is one. This shows that our law for Ξ_N holds true for $N = 1$ and $N = 2$.

Let us now suppose that it holds for a generic N; i.e., we have that a mixed $(N - 1)$-mode CM with $\aleph = 1$ can be put in a standard form specified by $N(N - 1)/2$ parameters. Now let us check what happens for a $(N + 1)$-mode pure state, i.e., for the reduced N-mode mixed state with symplectic rank equal to 1. A general way (up to LUs) of constructing a N-mode CM with $\aleph = 1$ yielding generic entanglement is the following: (a) take a generic-entangled $(N - 1)$-mode CM with $\aleph = 1$ in standard form; (b) append an ancillary mode (σ_N) initially in the vacuum state (the mode cannot be thermal as $\aleph$ must be preserved); (c) squeeze mode N with an arbitrary s (one has this freedom because it is a local symplectic operation); (d) let mode N interact couplewise with all the other modes, via a chain of beam splitters [13] with arbitrary transmittivities $b_{i,N}$, with $i = 1, \ldots, N - 1$ [14]; (e) if desired, terminate with N suitable single-mode squeezing operations (but with all squeezings now *fixed* by the respective reduced CM's elements) to symplectically diagonalize each single-mode CM.

With these steps one is able to construct a mixed state of N modes, with the desired rank, and with generic (LU-invariant) properties for each single-mode individual CM. We will show in the following that in the considered states the pairwise quantum correlations between any two modes are unconstrained. To conclude, let us observe that the constructed generic-entangled state is specified by a number of parameters equal to: $N(N - 1)/2$ [the parameters of the starting $(N - 1)$-mode mixed state of the same form] plus 1 (the initial squeezing of mode N) plus $N - 1$ (the two-mode beam-splitter interactions between mode N and each of the others). Total: $(N + 1)N/2 = \Xi_{N+1}$. ∎

Quantum state engineering.— Following the ideas of the above proof, a physically insightful scheme to produce generic-entangled N-mode pure Gaussian states can be readily presented (see Fig. 1). It consists of basically two main steps: (1) creation of the state in the $1|(N - 1)$ Schmidt decomposition and (2) addition of modes and entangling operations [13] between them. One starts with a chain of N vacua.

First of all (step 1), the recipe is to squeeze mode 1 of an amount s, and mode 2 of an amount $1/s$ (i.e., one squeezes the first mode in one quadrature and the second, of the same amount, in the orthogonal quadrature); then one lets the two modes interfere at a 50:50 beam splitter. One has so created a two-mode squeezed state between modes 1 and 2, which corresponds to the Schmidt form of σ with respect
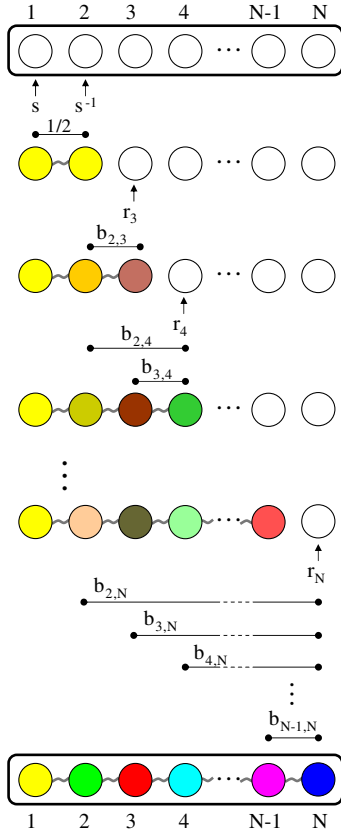
FIG. 1 (color online). How to create a generic-entangled *N*-mode pure Gaussian state. White balls are vacua, while each color depicts a different single-mode determinant (i.e., different degrees of local mixedness). Vertical arrows denote single-mode squeezing operations, while horizontal circle-ended lines denote beam-splitting operations $b_{i,j}$ between modes *i* and *j*. See text for details.

to the $1|(N-1)$ bipartition. The second step basically corresponds to create the most general mixed state with $\aleph = 1$, of modes $2, \ldots, N$, out of its Williamson diagonal form. This task can be obtained, as already sketched in the above proof, by letting each additional mode interact step by step with all the previous ones. Starting with mode 3 (which was in the vacuum like all the subsequent ones), one thus squeezes it (of an amount r_3) and combines it with mode 2 via a beam splitter (characterized by a transmittivity $b_{2,3}$). Then one squeezes mode 4 by r_4 and lets it interfere sequentially both with mode 2 (with transmittivity $b_{2,4}$) and with mode 3 (with transmittivity $b_{3,4}$). This process can be iterated for each other mode, as shown in Fig. 1, until the last mode *N* is squeezed (r_N) and entangled with the previous ones via beam splitters with respective transmittivities $b_{i,N}$, $i = 2, \ldots, N-1$. Step 2 describes the distribution of the two-mode entanglement created in step 1, among all modes.

The presented prescription enables to create a generic form (up to LUs) of multipartite entanglement among *N* modes in a pure Gaussian state, by means of active

(squeezers) and passive (beam splitters) linear optical elements. What is relevant for practical applications is that the state engineering is implemented with minimal resources. Namely, the process is characterized by one squeezing degree (step 1), plus $N - 2$ individual squeezings for step 2, together with $\sum_{i=1}^{N-2} i = (N-1)(N-2)/2$ beam-splitter transmittivities, which amount to a total of $N(N-1)/2 \equiv \Xi_N$ quantities. The optimally produced Gaussian states can be readily implemented for *N*-party CV communication networks [2,15].

Standard form.—The special subset of pure *N*-mode Gaussian states emerging from our constructive proof exhibits a distinct property: all correlations between "position" $\hat{q}_i$ and "momentum" $\hat{p}_j$ operators are vanishing. Looking at Eq. (1), this means that such a generic-entangled pure Gaussian state can be put in a *standard form* where all the 2×2 submatrices of its CM are diagonal. The diagonal subblocks σ_i can be additionally made proportional to the identity by local Williamson diagonalizations in the individual modes. This standard form for generic-entangled *N*-mode Gaussian states, as already mentioned, can be achieved by *all* pure Gaussian states for $N = 2$ [3] and $N = 3$ [4]; for $N \geq 4$, pure Gaussian states can exist whose number of independent parameters scales as $N(N-2)$ [5] and which cannot thus be brought in the $\hat{q}$-$\hat{p}$ block-diagonal form. Interestingly, all pure Gaussian states in our considered block-diagonal standard form are ground states of quadratic Hamiltonians with springlike interactions [16].

Vanishing $\hat{q}$-$\hat{p}$ covariances imply that the CM can be written as a direct sum $\sigma^p = \mathbf{V}_Q \oplus \mathbf{V}_P$, when the canonical operators are arranged as $\{\hat{q}_1, \ldots, \hat{q}_N, \hat{p}_1, \ldots, \hat{p}_N\}$. Moreover, the global purity of σ^p imposes $\mathbf{V}_P = \mathbf{V}_Q^{-1}$. Named $(\mathbf{V}_Q)_{ij} = v_{Q_{ij}}$ and $(\mathbf{V}_P)_{hk} = v_{P_{hk}}$, this means that each $v_{P_{hk}}$ is a function of the $\{v_{Q_{ij}}\}$'s. The additional *N* Williamson conditions $v_{P_{ii}} = v_{Q_{ii}}$ fix the diagonal elements of $\mathbf{V}_Q$. The standard form is thus completely specified by the off-diagonal elements of the symmetric $N \times N$ matrix $\mathbf{V}_Q$, which are, as expected, $N(N-1)/2 \equiv \Xi_N$. Proposition 1 acquires now a remarkable physical insight: the structural properties of the generic-entangled *N*-mode Gaussian states, and, in particular, their bipartite and multipartite *entanglement*, are completely specified (up to LUs) by the "two-point correlations" $v_{Q_{ij}} = \langle \hat{q}_i \hat{q}_j \rangle$ between any pair of modes. For instance, the entropy of entanglement between one mode (say *i*) and the remaining $N - 1$ modes, which is monotonic in Detσ_i [10], is completely specified by assigning all the pairwise correlations between mode *i* and any other mode $j \neq i$, as Det$\sigma_i = 1 - \sum_{j \neq i}Det\varepsilon_{ij}$. The rationale is that entanglement in such states is basically reducible to a mode-to-mode one. This statement, strictly speaking true only for the pure Gaussian states for which Proposition 1 holds, acquires a general validity in the context of the modewise decomposition of arbitrary pure Gaussian states [11,12]. This correlation picture

breaks down for mixed Gaussian states, where also classical, statistical-like correlations arise.

Gaussian matrix product states.—As an application, let us consider Gaussian matrix product states (GMPS), defined as N-mode states obtained by taking a fixed number, M, of infinitely entangled ancillary bonds [Einstein-Podolski-Rosen (EPR) pairs] shared by adjacent sites, and applying an arbitrary $2M \to 1$ Gaussian operation $\mathcal{P}^{[i]}$ on each site $i = 1, \ldots, N$. The projections $\mathcal{P}^{[i]}$ can be described in terms of isomorphic $(2M + 1)$-mode pure Gaussian states with CM $\gamma^{[i]}$, the *building blocks* [17].

It is conjectured that all pure N-mode Gaussian states can be described as GMPS. Here we provide a lower bound on the number M of ancillary bonds required to accomplish this task, as a function of N. We restrict to ground states of harmonic chains with springlike interactions. With a simple counting argument, the total number of parameters of the initial chain of building blocks should be at least equal to that of the target state, i.e., $N(2M + 1)(2M)/2 \geq N(N - 1)/2$ which means $M \geq \mathrm{IntPart}[(\sqrt{4N - 3} - 1)/4]$. This implies, for instance, that to describe generic states with at least $N > 7$ modes, a single EPR bond per site is no more enough (even though the simplest case of $M = 1$ yields interesting families of N-mode GMPS for any N [17]). The minimum M scales as $N^{1/2}$, diverging in the field limit $N \to \infty$. As infinitely many bonds would be necessary (and maybe not even sufficient) to describe generic infinite harmonic chains, the matrix product formalism is probably not helpful to prove or disprove area law statements for critical bosons (complementing the known results for the noncritical case [18]), which in general do not fall in special subclasses of finite-bonded GMPS.

The matrix product picture, however, effectively captures the entanglement distribution in translationally invariant N-mode harmonic rings [17]. In this case the GMPS building blocks are equal at all sites, $\gamma^{[i]} \equiv \gamma \forall i$, while the number of parameters Eq. (2) of the target state reduces to the number of independent pairwise correlations (only functions of the distance between the two sites), which by our counting argument is $\Theta_N \equiv (N - N \bmod 2)/2$. The corresponding threshold for a GMPS representation becomes $M \geq \mathrm{IntPart}[(\sqrt{8\Theta_N + 1} - 1)/4]$. As Θ_N is bigger for even N, so it is the resulting threshold, which means that in general a higher number of EPR bonds is needed, and so more entanglement is inputed in the GMPS projectors and gets distributed in the target N-mode Gaussian state, as opposed to the case of an odd N. This clarifies why nearest-neighbor entanglement in ground states of pure translationally invariant N-mode harmonic rings (which belong to the class of states characterized by Proposition 1) is frustrated for odd N [19].

Conclusions.—In this Letter we studied pure N-mode Gaussian states of CV systems, aimed to eradicate the minimal number of degrees of freedom responsible for the generic nonlocal features of the states. We showed that a crucial subclass of such states, employed as typical resources in CV quantum information, can be put in a standard form described (up to local unitaries) by $N(N - 1)/2$ parameters, corresponding to the ground states of a quadratic Hamiltonian with springlike couplings. This form encompasses *all* pure Gaussian states for $N \leq 3$. We operationally related these parameters with the active and passive transformations needed to prepare the state. In general, we interpreted those degrees of freedom as the two-point correlations between any pair of modes, which are thus responsible for the structure of *generic entanglement* in Gaussian states. It would be worth to investigate the exotic properties of multipartite entanglement arising in Gaussian states which *cannot* be brought in the standard form described here for $N \geq 4$.

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