Fractional Quantum Hall States of Dirac Electrons in Graphene

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We have investigated the fractional quantum Hall states of Dirac electrons in a graphene layer in different Landau levels. The relativistic nature of the energy dispersion relation of electrons in graphene significantly modifies the interelectron interactions. This results in a specific dependence of the ground state energy and the energy gaps for electrons on the Landau-level index. For the valley-polarized states, i.e., at $\nu = 1/m$, m being an odd integer, the energy gaps have the largest values in the $n = 1$ Landau level. For the valley-unpolarized states, e.g., for the $2/3$ state, the energy gaps are suppressed for $n = 1$ as compared to those at $n = 0$. For both $n = 1$ and $n = 0$, the ground state of the $2/3$ system is fully valley-unpolarized.

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A two-dimensional electron system in a single layer of graphite (graphene) is known to exhibit many remarkable properties. From the band structure studies [1], it was established early on that, to a good approximation, the energy dispersion of electrons in graphene is *linear* near the points at the corners of the Brillouin zone where the valence band and the conduction band meet. As a consequence, the low-energy excitations follow the Dirac-Weyl equations for massless *relativistic* particles [2]. In an external magnetic field, the electron system also shows unique properties that are different from those of the standard *nonrelativistic* electron systems [2–4]. Recent experimental demonstration of some of those properties, in particular, the discovery of the integer quantum Hall effect [5,6] that was predicted in earlier theoretical works [7], has caused intense interest in the electronic properties of the Dirac electrons in graphene [8]. However, effects of electron correlations in this system have not been reported yet. In this Letter, we report on the nature of the fractional quantum Hall states of Dirac electrons in graphene.

A unit cell of the two-dimensional (2D) graphene honeycomb lattice contains two carbon atoms, say, A and B. The dynamics of electrons in graphene is described by a tight-binding Hamiltonian with the nearest-neighbor hopping. In the continuum limit, this Hamiltonian generates the band structure with two π bands, and the Fermi levels are located at two inequivalent points, $K = (2\pi/a) \times (1/3, 1/\sqrt{3})$ and $K' = (2\pi/a)(2/3, 0)$, of the first Brillouin zone, where $a = 0.246$ nm is the lattice constant. Near the points K and K', the electrons have a linear Dirac-Weyl ("relativistic") type dispersion relation [1,2]. Finally, in the continuum limit, the electron wave function is described by the 8-component spinor $\Psi_{s,k,\alpha}$, where $s = \pm 1/2$ is the spin index, $k = K, K'$ is the valley index, and $\alpha = A, B$ is the sublattice index. Without the spin-orbit interaction [9–11], the Hamiltonian is described by two 4×4 matrices for each component of the electron spin. In the presence of a magnetic field perpendicular to the graphene plane, the Hamiltonian matrix has the form

$$\mathcal{H} = \frac{\gamma}{\hbar}\begin{pmatrix} 0 & \pi_x - i\pi_y & 0 & 0 \\ \pi_x + i\pi_y & 0 & 0 & 0 \\ 0 & 0 & 0 & \pi_x + i\pi_y \\ 0 & 0 & \pi_x - i\pi_y & 0 \end{pmatrix}, \tag{1}$$

where $\vec{\pi} = \vec{p} + e\vec{A}/c$, $\vec{p}$ is the two-dimensional momentum, $\vec{A}$ is the vector potential, and γ is the band parameter. The ordering used for the basis states in the noninteracting Hamiltonian is $(K, A; K, B; K', A; K', B)$. The eigenfunctions of the Hamiltonian are specified by the Landau-level index $n = 0, \pm 1, \pm 2, \ldots$ and the intra-Landau-level index m that is gauge dependent. Each Landau level is fourfold degenerate due to the spin and valley degrees of freedom. The corresponding wave functions for an electron in the two valleys K and K' are described by

$$\Psi_{K,n} = C_n \begin{pmatrix} \mathrm{sgn}(n) i^{|n|-1}\phi_{|n|-1} \\ i^{|n|}\phi_{|n|} \\ 0 \\ 0 \end{pmatrix}, \tag{2}$$

$$\Psi_{K',n} = C_n \begin{pmatrix} 0 \\ 0 \\ i^{|n|}\phi_{|n|} \\ \mathrm{sgn}(n) i^{|n|-1}\phi_{|n|-1} \end{pmatrix}, \tag{3}$$

where $C_n = 1$ for $n = 0$ and $C_n = 1/\sqrt{2}$ for $n \neq 0$. Here ϕ_n is the standard Landau wave function for a particle with nonrelativistic parabolic dispersion relation in the nth Landau level. From Eqs. (2) and (3), it is clear that a specific feature of the relativistic dispersion law is the mixing of the nonrelativistic Landau levels. This mixture is present only for $n \neq 0$ and strongly modifies the interelectron interaction within a single Landau level.

In what follows, we study the partially occupied Landau levels with fractional filling factors. Partial occupation of the Landau levels is realized by doping of the graphene layer. Experimentally, different filling factors of the Landau levels are achieved by varying the applied magnetic field at a fixed electron concentration. In this case, the ground state of the system and the excitation spectrum are fully determined by the interelectron interactions. For the nonrelativistic case, this results in the incompressible fractional quantum Hall effect (FQHE) states at the fractional filling factors [12,13]. Properties of these states are completely described by Haldane's pseudopotentials V_m [14], which are the energies of two electrons with relative angular momentum m. The pseudopotentials for the nth Landau level can be presented as

$$V_m^{(n)} = \int_0^\infty \frac{dq}{2\pi} qV(q)[F_n(q)]^2 L_m(q^2) e^{-q^2}, \qquad (4)$$

where $L_m(x)$ are the Laguerre polynomials, $V(q) = 2\pi e^2/(\kappa l q)$ is the Coulomb interaction in the momentum space, κ is the dielectric constant, l is the magnetic length, and $F_n(q)$ is the form factor corresponding to the nth Landau level. For relativistic electrons, the form factor is given by the expression [15]

$$F_0(q) = L_0\left(\frac{q^2}{2}\right), \qquad (5)$$

$$F_{n\neq 0}(q) = \frac{1}{2}\left[L_n\left(\frac{q^2}{2}\right) + L_{n-1}\left(\frac{q^2}{2}\right)\right], \qquad (6)$$

while for the nonrelativistic particles the form factors in Eq. (4) are $F_n(q) = L_n(q^2/2)$. This means that the interelectron interactions for the relativistic and nonrelativistic electrons are the same for $n = 0$ and are different for $n > 0$ [15]. In what follows, all energies are expressed in units of the Coulomb energy $\varepsilon_c = e^2/\kappa l$.

In Fig. 1, we compare the pseudopotentials calculated from Eq. (4) for the relativistic and the nonrelativistic cases. For the relativistic electrons [Fig. 1(a)], we notice a clear suppression of the pseudopotential for $n = 1$ as compared to that at $n = 0$, only for $m = 0$, i.e., when both electrons are at the same spatial point. For all other m, we have the inequality $V_m^{(1)} > V_m^{(0)}$. This is different from the nonrelativistic case, where the pseudopotential is suppressed also for $m = 1$, i.e., $V_1^{(1)} < V_1^{(0)}$. We also see in Fig. 1(a) that, although the relativistic wave function at $n = 1$ is the "mixture" of the $n = 0$ and $n = 1$ nonrelativistic wave functions, the relativistic pseudopotential is not the average of the corresponding nonrelativistic pseudopotentials. This is clearly seen for $m = 1$, where the relativistic pseudopotential at $n = 1$ is larger than the nonrelativistic one for both $n = 0$ and $n = 1$. In Fig. 1(b), the relativistic pseudopotentials are shown for different Landau levels. Here the special case is $m = 1$, where the dependence of the pseudopotential on the Landau-level index is nonmonotonic; viz., the pseudopotential has the maximum value at $n = 1$. At all of the other m values, the trend is the same as for the nonrelativistic case; i.e., for $m = 0$ the pseudopotential decreases with increasing n, while at $m > 1$ the pseudopotentials increase with n.

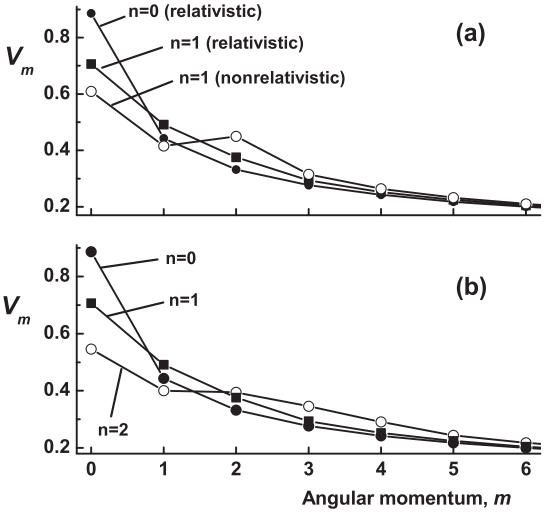

FIG. 1. The pseudopotentials [Eq. (4)] as a function of the relative angular momentum (a) for relativistic and for nonrelativistic 2D electrons in the first two Landau levels and (b) for relativistic electrons in various Landau levels.

With the pseudopotentials for Dirac electrons at hand, we now evaluate the energy spectra of the many-electron states at fractional fillings of the Landau level. The calculations have been done in the spherical geometry [14] with the pseudopotentials given by Eq. (4). The radius of the sphere R is related to $2S$ of magnetic fluxes through the sphere in units of the flux quanta as $R = \sqrt{S}l$. The single-electron states are characterized by the angular momentum S and its z component S_z. For a given number of electrons N, the parameter S determines the filling factor of the Landau level. Because of the spherical symmetry of the problem, the many-particle states are described by the total angular momentum L and its z component, while the energy depends only on L. At first, we study the system with the fractional filling factor $\nu = 1/3$, which corresponds to the $1/3$ FQHE. In the spherical geometry, the $1/3$-FQHE state is realized at $S = (3/2)(N - 1)$. If the electron system is fully spin- and valley-polarized, then we should expect the ground state to be in the Laughlin state [16], which is separated from the excited states by a finite gap. We calculated the energy spectra of a finite-size system by finding the lowest eigenvalues and eigenvectors of the interaction Hamiltonian matrix [17]. In these calculations, we take into account the interaction between all of the electrons of the partially occupied Landau levels. We have also addressed the question of polarization of the many-particle state. We assume that at a high magnetic field the system is always spin-polarized. In this case, the system can be either valley-polarized or valley-unpolarized. Similar to the spin polarization of the standard FQHE states, the valley polarization of the graphene system depends on the intervalley asymmetry (same as the Zeeman splitting for the nonrelativistic system) and filling factor of the Landau level. Namely, if the intervalley splitting due to the intervalley asymmetry is large, then the system is always valley-polarized, while otherwise the valley polarization of the system depends on the filling factor of the Landau level.

In Fig. 2, we show the calculated energy spectra for the $1/3$-FQHE state at different Landau levels. Here the $1/3$-FQHE state at the nth Landau level is defined as the state corresponding to the $1/3$ filling factor (single valley, single spin) of the nth Landau level, while all the lower energy Landau levels are completely occupied. Since the relativistic pseudopotential $V_m^{(0)}$ for $n = 0$ is the same as for the nonrelativistic one, the $1/3$ state and the corresponding energy gap (in units of ε_c) will be the same in both cases. The deviation from the nonrelativistic system occurs only at higher Landau levels. In Fig. 2(a), the energy gap of the $1/3$ state at $n = 1$ is noticeably enhanced compared to that at $n = 0$. This is a direct manifestation of the specific dependence of pseudopotenials $V_m^{(n)}$ on the Landau-level index. Because of the asymmetry of the electron wave functions, the spectra of the $1/3$-FQHE state is determined mainly by the relative value of $V_1^{(n)}$ and $V_3^{(n)}$ pseudopotentials, which have the highest value at $n = 1$. The energy spectra of the $1/3$-FQHE state at $n = 2$, shown in Fig. 2(b), demonstrate a strong suppression of the gap when compared to the $n = 1$ and $n = 0$ FQHE states. We therefore conclude that *the* $1/3$-FQHE *state in graphene is most stable at* $n = 1$. Hence, the interelectron interaction effects are more pronounced at $n = 1$. Interestingly, this tendency is just the opposite of that of the nonrelativistic system, where the excitation gap decreases monotonically with increasing Landau-level index [12].

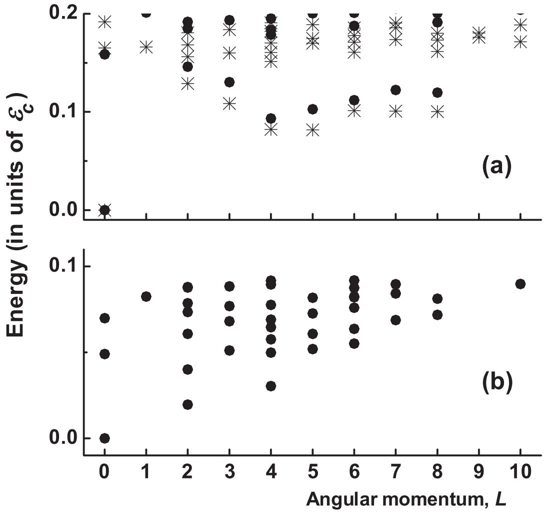

FIG. 2. Energy spectra of the eight-electron $\nu = 1/3$-FQHE system, shown for different Landau levels: (a) $n = 0$ (stars), $n = 1$ (dots), and (b) $n = 2$. The system is fully spin- and valley-polarized. The flux quanta is $2S = 21$.

The results in Fig. 2 correspond to a completely spin- and valley-polarized system. This polarization is achieved at a high magnetic field due to the Zeeman splitting and the valley asymmetry. The intervalley asymmetry has two sources: The first one is due to interaction-induced "backscattering" between different valleys [15], while the second one is due to the asymmetry in the lattice-scale interactions within the two sublattices of graphene [18]. Since the positions of the electrons in two sublattices are shifted, the interaction between the electrons in the different sublattices is weaker than the interaction between the electrons in the same sublattice. Both effects vary as (a/l), so they become more relevant at higher magnetic fields or at a smaller magnetic length.

The same picture holds for the other FQHE states of the type $1/m$; i.e., the state is most stable at $n = 1$. The new aspects of the interaction physics arise at other filling factors as well, when the lowest energy states are spin-unpolarized for the nonrelativistic single-valley systems. The simplest example is $\nu = 2/3$. In this case, the ground state of the nonrelativistic electrons is spin-unpolarized at a small Zeeman splitting. The transition from the spin-polarized to the spin-unpolarized ground states in a tilted magnetic field is well established both experimentally and theoretically for nonrelativistic electrons [13].

Here we address the problem of the polarized and unpolarized states in the graphene system. We assume that the Zeeman splitting is large enough so that all of the states are spin-polarized. At the same time, the intervalley asymmetry is small and electrons can occupy both of the valleys. In this case, the valley-polarized and valley-upolarized $2/3$-FQHE states become relevant. For a valley-polarized system, the excitation spectra and the ground state properties of the $2/3$ state are the same as those for $\nu = 1/3$ due to the particle-hole symmetry. Similar to the $1/3$ case, we obtain numerically an enhancement of the excitation gap at $n = 1$ as compared to that at $n = 0$. The more complicated situation occurs for the valley-unpolarized system. First, we compare, just as for the nonrelativistic system [19], the energy of the ground states of the polarized and the unpolarized systems. In the spherical geometry, the polarized and the unpolarized states are realized for different sizes of the sphere, i.e., for different flux quanta through the sphere. For the $2/3$-FQHE system, the polarized state occurs at $2S = 3N/2$, while the unpolarized one is at $2S = 3N/2 - 1$. Because of the different size of the sphere in these two systems, the finite-size corrections to the magnetic length should be introduced, $l' = (\nu 2S/N)^{1/2} l$ [19,20].

We have calculated the ground state energies for the valley-polarized and the valley-unpolarized graphene sys-

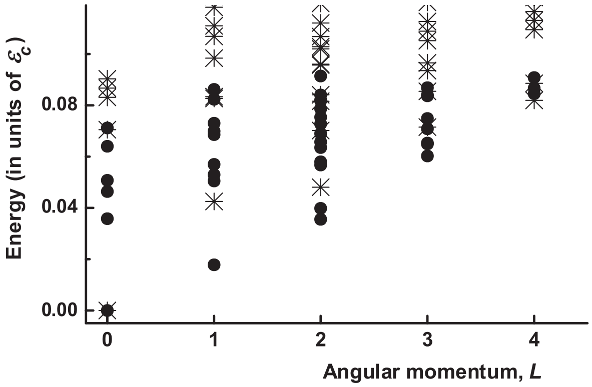

FIG. 3. Energy spectra of the eight-electron $\nu = 2/3$-FQHE system, shown for different Landau levels: $n = 0$ (stars) and $n = 1$ (dots). The system is valley-unpolarized and fully spin-polarized. The flux quanta is $2S = 11$.

tem in the $n = 1$ and the $n = 0$ Landau levels for an eight-electron system in the spherical geometry. We found that for both $n = 0$ and $n = 1$ the valley-unpolarized state has the lower energy. The ground state energy per particle in the unpolarized system is lower than that for the polarized system by $0.073\varepsilon_c$ for $n = 0$ and by $0.053\varepsilon_c$ for $n = 1$. Therefore, the valley-unpolarized state is more favorable for $n = 0$. Here the gap between the polarized and the unpolarized states is suppressed for $n = 1$ as compared to the $n = 0$ case. This is opposite to the completely polarized system (Fig. 2), where the effects of interaction are the strongest for $n = 1$. Suppression of the interaction effects in an unpolarized system for $n = 1$ is also illustrated in Fig. 3, where the excitation spectra of the valley-unpolarized system are shown for $n = 0$ and $n = 1$. A strong suppression of the excitation gaps for $n = 1$ is clearly visible here. The origin of this suppression can be understood from the dependence of the pseudopotentials $V_m^{(n)}$ on the relative angular momentum m in different Landau levels. Because of the Pauli exclusion principle, the energetic properties of the polarized state are determined only by the pseudopotentials with odd angular momenta $m = 1, 3, 5, \ldots$. These pseudopotentials have the largest values for $n = 1$, which results in the strongest interaction effects for $n = 1$, in a polarized system. For an unpolarized system, the properties of the ground and excited states depend on all of the pseudopotentials. Since $V_m^{(n)}$ at $m = 0$ is strongly suppressed for $n = 1$ as compared to the $n = 0$ case, we expect a suppression of interaction effects in an unpolarized system in the $n = 1$ Landau level.

In conclusion, the relativistic nature of the energy dispersion of electrons in the graphene plane modifies the interelectron interactions significantly. This results in a unique dependence of the ground state energy and the energy gaps of the graphene systems on the Landau-level index. For the valley-polarized states, i.e., at $\nu = 1/m$, the FQHE gaps have the largest values for $n = 1$. Based on these studies, we conclude that the FQHE at $\nu = 1/m$ should be observed experimentally for both $n = 0$ and $n = 1$, perhaps in a higher mobility system. For the valley-unpolarized states, e.g., for the 2/3-FQHE state, the energy gaps are suppressed at $n = 1$ as compared to that for the $n = 0$ level. For both $n = 1$ and $n = 0$, the ground state of the 2/3-FQHE system is fully unpolarized. The intervalley asymmetry will result in transitions between the valley-polarized and the valley-unpolarized states.

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