Chiral Phase Transition from String Theory

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The low energy dynamics of a certain *D*-brane configuration in string theory is described at weak t'Hooft coupling by a nonlocal version of the Nambu-Jona-Lasinio model. We study this system at finite temperature and strong t'Hooft coupling, using the string theory dual. We show that for sufficiently low temperatures chiral symmetry is broken, while for temperatures larger then the critical value, it gets restored. We compute the latent heat and observe that the phase transition is of the first order.

Introduction.—Over the past few years there has been considerable progress in studying QCD-like theories using holography [an incomplete list of references includes [1–7]]. The gravity description is useful for studying properties of the hadrons as well as qualitative features of QCD such as confinement and chiral symmetry breaking (χSB). The latter stands for the dynamical breaking of the $U(N_f) \times U(N_f)$ symmetry acting on N_f right-handed and left-handed quarks of the high energy Lagrangian down to its diagonal subgroup, $U(N_f)$. In QCD the energy scales of confinement and χSB are approximately the same.

An interesting example which exhibits chiral symmetry breaking is the Nambu-Jona-Lasinio (NJL) model [8], whose Lagrangian contains left- and right-handed quarks interacting via a nonrenormalizable quartic term. The nonrenormalizability of this interaction makes it natural to attempt a well-defined UV completion. Recently, such a model was proposed within the context of string theory [9]. The weak coupling string theory description of the model considered in [9] involves a set of N_f *D*8 branes and N_f anti-*D*8 branes separated by a distance L in the x^4 direction and N_c *D*4 branes extended in the $(x^0, \ldots, x^4)$ directions. At low energies, the system reduces to the N_f left-handed quarks and N_f right-handed quarks interacting via a five-dimensional gluon exchange. This induces a nonlocal Nambu-Jona-Lasinio-type interaction in the low energy Lagrangian for the quarks.

An important feature of the large N_c model studied in [9] is the separation of scales. Namely, the energy scale associated with confinement is well below that of chiral symmetry breaking. In addition, by compactifying the x^4 direction and varying the radius of the resulting circle one can smoothly interpolate between the NJL model and QCD. The weakly coupled region of the former can be analyzed within field theory [9]. At strong coupling, one can use holographically dual string theory. Chiral symmetry breaking and meson properties can be analyzed within the DBI action for the N_f *D*8 branes living in the background created by N_c *D*4 branes. As long as $N_f \ll N_c$, backreaction of the *D*8 brane on the background metric can be ignored. The conclusion of [9] is that at strong t'Hooft

coupling chiral symmetry is dynamically broken, since the corresponding brane configuration is energetically favored over the one with unbroken symmetry.

In this Letter we study the effects of finite temperature in the limit of strong t'Hooft coupling. We find that there is a phase transition at the temperature $T_c \approx 0.15 \, \mathrm{L}^{-1}$. Below this temperature, there exists a thermodynamically favorable brane configuration where chiral symmetry is dynamically broken. As we increase the temperature above the T_c, we find that the solution with the unbroken chiral symmetry is thermodynamically preferred. As temperature reaches $T_* \approx 0.17 \, \mathrm{L}^{-1}$ the phase with broken chiral symmetry ceases to exist. These properties suggest that the phase transition is of the first order. To confirm this we compute the latent heat, and show that it is positive. The rest of the Letter is organized as follows. In the next section we introduce the brane model at finite temperature and discuss the features of the phase transition at $T = T_c$. We discuss our results in Section 3.

*D*8 *and D*4 *branes at finite temperature.*—We will consider the near-horizon geometry of the *D*4 branes in the limit studied in [9], where the direction transverse to the *D*8 branes is noncompact. The near-horizon geometry of the *D*4 branes at finite temperature is

$$ds^2 = \left(\frac{U}{R}\right)^{3/2}[dx_i dx^i + f(U)dt^2 + (dx^4)^2]$$
$$+ \left(\frac{U}{R}\right)^{-3/2}\left(\frac{dU^2}{f(U)} + U^2 d\Omega_4^2\right), \quad (1)$$

where $f(U) = 1 - U_T^3/U^3$. Here t is the Euclidean time, $i = 1, \ldots, 3$; U and Ω_4 label the radial and angular directions of $(x^5, \ldots, x^9)$. The parameter R is given by

$$R^3 = \pi g_s N_c l_s^3 = \pi \lambda, \quad (2)$$

where in the last equality and in the rest of the Letter we set $l_s = 1$. The four-brane geometry also has a nontrivial dilaton background,

$$e^\Phi = g_s \left(\frac{U}{R}\right)^{3/4}. \quad (3)$$

Finite temperature implies that t is a periodic variable,

$$t \sim t + \beta. \qquad (4)$$

On the other hand, in order for (1) to describe a nonsingular space, t must satisfy

$$t \sim t + \frac{4\pi R^{3/2}}{3U_T^{1/2}}. \qquad (5)$$

Hence, the temperature is related to the minimal value of U, denoted by U_T, as $T = 3U_T^{1/2}/4\pi R^{3/2}$.

Quarks can be incorporated into this system by adding N_f $D8 - \bar{D}8$ brane pairs. In the following we consider $N_f = 1$ case for simplicity, but the discussion applies for finite $N_f \ll N_c$ as well. At weak coupling, $D8$ and $\bar{D}8$ branes are separated by the distance L in the x^4 direction. At strong coupling $\lambda \gg L, l_s$, the shape of $D8 - \bar{D}8$ brane pair is determined by the equation of motion that follows from the DBI action. As we will see below, there are two possible types of solutions, straight and curved. The straight solution is simply $\tau = $ const, and describes separated $D8$ and $\bar{D}8$ branes. In this situation, chiral symmetry is not broken. In the Lorentzian version of (1), the point $U = U_T$ corresponds to the horizon of the black hole. The straight branes therefore cross the horizon in the Lorentzian space. The second type of solution is the one where the $D8 - \bar{D}8$ brane surface is connected and is described by a curve $U(x_4)$ in the geometry (1). This configuration breaks chiral symmetry and does not cross the horizon in the Lorentzian space. To understand which configuration is thermodynamically preferred, it is necessary to compare the values of the free energy, which we do below.

In the following it will be convenient to parametrize the $D8 - \bar{D}8$ brane surface by the embedding function $\tau(x^\mu, U) \equiv x^4(x^\mu, U)$. The classical trajectory is x^μ independent, and $U \to \infty$ as $\tau \to \pm L/2$ [9]. The D brane has two branches which are interchanged by $\tau \to -\tau$. We will mostly restrict attention to one branch, where U varies from ∞ to the limiting value U_0 at $\tau = 0$. The induced metric on the brane in the background (1) is

$$ds^2 = \left(\frac{U}{R}\right)^{3/2}[(\eta_{\mu\nu} + \partial_\mu\tau\partial_\nu\tau)dx_\mu dx^\nu + [f(U) - 1]dt^2]$$
$$+ 2\left(\frac{U}{R}\right)^{3/2}\partial_U\tau\partial_\mu\tau dx^\mu dU$$
$$+ \left(\frac{U}{R}\right)^{-3/2}\left[\frac{1}{f(U)} + \left(\frac{U}{R}\right)(\partial_U\tau)^2\right]dU^2$$
$$+ R^{3/2}U^{1/2}d\Omega_4^2, \qquad (6)$$

where $\mu = 0, \ldots, 3$. At this point we assume that the brane configuration is independent of x^μ. The DBI action is

$$S = \frac{T_8 V_S R^{3/2}}{T} \int d^3x \int dU U^{5/2}\sqrt{1 + (U/R)^3 f(U)(\partial_U\tau)^2}, \qquad (7)$$

where $T_8 = 1/(2\pi)^8 g_s$ is the tension of the brane and

$V_S = \pi^2/2$ is the volume of the unit four-sphere. It is convenient to make a change of variables

$$y = \frac{2R^{3/2}}{\sqrt{U}}, \qquad U = \frac{4R^3}{y^2} \qquad (8)$$

which brings (7) to the form

$$S = \frac{2^6\pi^2 T_8 R^{12}}{T} \int d^3x \int dy y^{-8}\sqrt{1 + f(y)(\partial_y\tau)^2}, \qquad (9)$$

where

$$f(y) = 1 - \frac{y^6}{y_T^6}, \qquad y_T = \frac{3}{2\pi T}. \qquad (10)$$

The space is restricted to lie in the region $y \in (0, y_T)$, whose upper limit corresponds to the black hole horizon which shrinks to a point after the Wick rotation. The equation of motion in the new variables is

$$\partial_y\left[\frac{y^{-8}f(y)\partial_y\tau}{\sqrt{1 + f(\partial_y\tau)^2}}\right] = 0. \qquad (11)$$

We denote conserved quantity in the square brackets by y_0^{-8}; this parameter parametrizes the solution. Then, (11) can be written in the form

$$\left(\frac{d\tau}{dy}\right)^2 = \frac{1}{f(y)[f(y)y_0^{16}/y^{16} - 1]}. \qquad (12)$$

Note that the brane extends all the way from $y = 0$ (which corresponds to the asymptotic region, $U \to \infty$) to $y = y_*$ which is determined by

$$\left(1 - \frac{y_*^6}{y_T^6}\right)\frac{y_0^{16}}{y_*^{16}} - 1 = 0. \qquad (13)$$

In particular, for $y_T \gg y_0$, $y_* \approx y_0$, while in the opposite regime $y_T \ll y_0$,

$$y_* = y_T\left[1 - \frac{y_T^{16}}{6y_0^{16}} + \mathcal{O}(y_T/y_0)^{32}\right]. \qquad (14)$$

In the limit $y_0 \to \infty$ the D-brane solution gets closer and closer to the horizon.

In the discussion above y_0 is a constant which parametrizes the solutions of equation of motion. Its relation to the physical parameters $y_T = 3/2\pi T$ and L is determined by

$$\frac{L}{2} = \int_0^{y_*} \frac{dy}{\sqrt{f(y)[f(y)y_0^{16}/y^{16} - 1]}}$$
$$= y_T \int_0^{x_*} \frac{dx}{\sqrt{f(x)[f(x)x_0^{16}/x^{16} - 1]}}, \qquad (15)$$

where we introduced the rescaled variable $x = y/y_T$. Analogously, $x_0 = y_0/y_T$ and $x_* = y_*/y_T$. In the limit of small y_0, L differs from it by a multiplicative factor of order unity,

$$L \approx y_0 \frac{B(9/16, 1/2)}{8}, \qquad y_0 \ll y_T. \tag{16}$$

In the opposite regime we need to estimate the integral,

$$\frac{L}{2} \approx y_T \int_0^{x_*} \frac{dx}{\sqrt{(1 - x^6)[(1 - x^6)x^{16}/x_0^{16} - 1]}}, \quad x_0 \to \infty. \tag{17}$$

To estimate the behavior of the integral in (17) at large x_0 it is convenient to introduce a new variable

$$z = (1 - x^6) - \frac{x^{16}}{x_0^{16}} \tag{18}$$

omitting terms subleading in $1/x_0^{16}$ we obtain

$$L \approx \frac{y_T}{3} \int_0^1 \frac{dz}{x^5 \sqrt{z}\sqrt{x^{16} + x_0^{16}z}} \sim y_T \left(\frac{y_T}{y_0}\right)^8 \tag{19}$$

which goes rapidly to zero as $y_0 \to \infty$.

We are thus led to the following picture. The integral over x in (15) goes like x_0 for small x_0 and peaks around $x_0 \sim 1$. It then goes down and goes rapidly to zero as $x_0 \to \infty$. Hence, for a sufficiently small temperature $T \ll 1/L$ we generically have two solutions for a $D8$ brane which asymptotes to $x^4 = \pm L/2$. This picture is confirmed by the numerical evaluation of the integral in (15) shown in Fig. 1.

For a sufficiently high temperature, there are no curved solutions, although there still exists a solution with constant x^4. In Minkowski space, this solution describes two collections of branes and antibranes which cross the horizon. For this configuration chiral symmetry is restored, which is what we expect from the high temperature phase. To understand the details of the chiral symmetry restoration, we need to compare the values of the action for the solutions described above.

The difference between the values of the free energy for the straight and curved solutions is

$$\frac{1}{T}(\mathcal{F}_{\text{straight}} - \mathcal{F}_{\text{curved}}) = \delta S = S_{\text{straight}} - S_{\text{curved}} \propto \int_{y_*}^{y_T} dy(y^{-8} - 0) + \int_0^{y_*} dy y^{-8}\left[1 - \left(1 - \frac{y^{16}}{y_0^{16}(1 - y^6/y_T^6)} \right)^{-1/2} \right]. \tag{20}$$

When the temperature is small, $y_T \gg L$, it is easy to consider the branch with $y_0 \sim L$. There, the result is the same as in [9],

$$\delta S \sim y_0^{-7} \sim y_T^{-7}(y_T/L)^7. \tag{21}$$

In particular, it is positive, i.e., the curved solution is preferred and chiral symmetry is unbroken. Moreover, δS blows up as $L \to 0$.

Consider now the second branch. Now the integral in (20) is dominated near the region $y \approx y_*$. To see this, introduce a new variable z defined by (18). The relevant part of the integral in (20) is

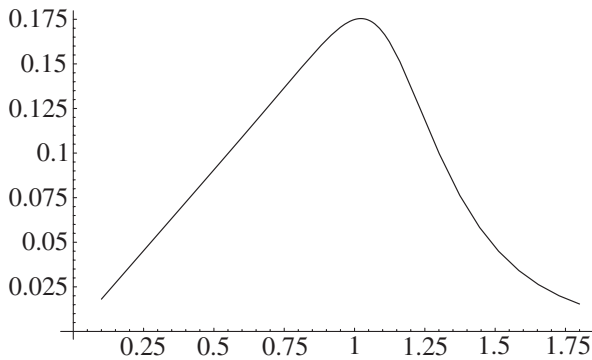$$\int_0^1 \left(1 - \sqrt{1 + \frac{1}{x_0^{16}}} \right) \sim -x_0^{-16} \log x_0, \tag{22}$$

where we neglect terms that are $\mathcal{O}(x_0^{-16})$. The result therefore is negative, and goes to zero as $L \to 0$:

$$\delta S \sim y_T^{-7}(y_T/y_0)^{-16} \log(y_0/y_T) \sim -y_T^{-7}(L/y_T)^2 \log, \tag{23}$$

where we used (19). In fact, as we see on Fig. 2, the difference (20) is positive for sufficiently small y_0, but then becomes negative.

The picture therefore is the following. When the temperature is low compared to $1/L$, the system resides in the state where chiral symmetry is broken and $y_0 \sim L$. This state is described by the graph in Fig. 1 with $x_0 \ll 1$. From Fig. 2 it is clear that the action for this branch is smaller then that for the straight brane solution, where chiral
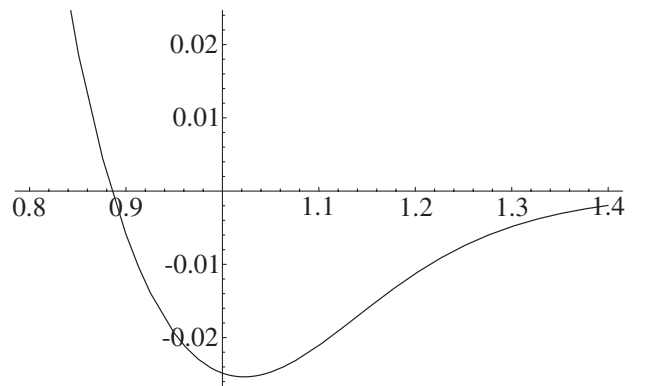

FIG. 1. The value of $L/2$ in the units of y_T [Eq. (15)] as a function of $x_0 = y_0/y_T$.

FIG. 2. The value of δS in appropriate units [Eq. (20)] as a function of $x_0 = y_0/y_T$.

symmetry is restored. It is interesting that there exists a second solution, with large y_0 for a sufficiently low temperature. This solution also breaks chiral symmetry, but is not thermodynamically preferred.

When the temperature is raised above the critical temperature $T_c \approx 0.15 \, L^{-1}$, the system undergoes a phase transition: the straight brane becomes thermodynamically preferred, and chiral symmetry gets restored. [The critical temperature corresponds to $x_0 \approx 0.885$, according to Fig. 2. This translates into $L/2y_T \approx 0.161$, according to Fig. 1. Using (10) gives the value of T_c quoted above.] As the temperature is increased past $T_* \approx 0.17 \, L^{-1}$, the only available solution is a straight brane. This picture suggests that the transition is of first order. Indeed it is easy to see that the latent heat is nonzero at $T = T_c$. The simplest way to see this is to note that the entropy density jumps at the transition

$$\frac{\Delta \mathcal{S}}{V_3} = -\frac{1}{V_3} \frac{\partial(\mathcal{F}_{\text{straight}} - \mathcal{F}_{\text{curved}})}{\partial T}. \tag{24}$$

Using (20), and computing the derivative at $T = T_c$ we find

$$\frac{\Delta \mathcal{S}}{V_3} \simeq 2.16 T_8 V_S \lambda^4 \left(\frac{4\pi}{3}\right)^7 T_c^6. \tag{25}$$

The latent heat per unit volume is positive and equal to

$$C_1 = \frac{T_c(\mathcal{S}_{\text{straight}} - \mathcal{S}_{\text{curved}})}{V_3} \simeq 0.03 \pi^5 N_c \lambda^3 T_c^7. \tag{26}$$

Discussion.—In this Letter we considered the string theory description of the chiral symmetry phase transition. The corresponding *D*-brane system at weak coupling is described by a certain NJL model with nonlocal interaction of the quarks. We argued that when the system is strongly coupled, $\lambda \gg L$, the following scenario takes place. As the temperature is low compared to the scale set by the brane separation, $1/L$, the curved $D8$ brane solution which breaks chiral symmetry is thermodynamically preferred over the straight $D8$ brane solution, with unbroken chiral symmetry. This solution goes over smoothly to the zero temperature solution studied in [9]. In particular, the maximal value of y on the $D8$ brane trajectory, denoted y_* in the Letter, scales as $y_* \sim L$ for small L, i.e., the D brane is getting further away from the horizon at $y = y_T$ as L decreases. As the temperature approaches $T_c \approx 0.15 \, L^{-1}$ from below, the free energy of the straight brane solution approaches that of the curved one. At $T = T_c$ the derivative of the free energy with respect to the temperature jumps, which indicates the first order phase transition. When the temperature is raised above $T_c \approx 0.15 \, L^{-1}$, the straight brane solution, which restores chiral symmetry, becomes thermodynamically preferred. For temperatures higher

then $T_* \approx 0.17 \, L^{-1}$, the straight brane is the only available solution. At low temperatures there also exists a second curved $D8$ brane solution, which comes asymptotically close to the horizon as $L \rightarrow 0$. The value of the action for this solution is higher then that of the other two.

To reiterate, the scale of the temperature where chiral phase transition happens is set by the asymptotic separation of the $D8$ branes L. It is interesting that this energy scale is much larger then that of Hawking-Page transition which is associated with confinement or deconfinement transition in field theory. To understand this better, consider the space (1) where the x^4 direction is compactified on a circle of radius R_4 [5]. In this case, the scale of Hawking-Page transition is set by $y_T \sim R_4$. The case of $R_4 \sim L$ corresponds to QCD, where both confinement and chiral symmetry breaking happen at the same scale. On the other hand, in the case $R_4 \gg L$, where our analysis applies, the two scales are widely separated, as has been also argued in [9].

There are many open questions. It would be interesting to generalize the discussion of this Letter for a nonzero chemical potential. There one expects nontrivial phase structure in the (μ, T) plane. Understanding the behavior of the mesons at finite temperature is another direction.

Note added.—Recently, we learned of the paper by O. Aharony, C. Sonnenschein, and S. Yankielowicz which partially overlaps with our results.

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