Splitting Times of Doubly Quantized Vortices in Dilute Bose-Einstein Condensates

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Recently, the splitting of a topologically created doubly quantized vortex into two singly quantized vortices was experimentally investigated in dilute atomic cigar-shaped Bose-Einstein condensates [Y. Shin *et al.*, Phys. Rev. Lett. **93**, 160406 (2004)]. In particular, the dependency of the splitting time on the peak particle density was studied. We present results of theoretical simulations which closely mimic the experimental setup. We show that the combination of gravitational sag and time dependency of the trapping potential alone suffices to split the doubly quantized vortex in time scales which are in good agreement with the experiments.

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Bose-Einstein condensation is associated with long-range phase coherence among the condensed particles. Because of this coherence, the dynamics of pure scalar condensates can be accurately described using only a single complex valued order parameter. The fact that the condensate current density is proportional to the phase gradient of the order parameter implies vorticity to be quantized: vortices are topological defects in the order parameter field with a quantum number multiple κ of 2π phase winding about the vortex line.

Especially, properties of quantized vortices in gaseous atomic Bose-Einstein condensates (BECs) have been intensively investigated during recent years, both experimentally and theoretically; see Ref. [1] for a review. Since the angular momentum associated with a vortex is roughly proportional to the vorticity quantum number κ but the energy to its square, multiquantum vortices are typically energetically unfavorable. Consequently, the usual experimental methods to create vortices have yielded only singly quantized vortices or clusters of them [2]. However, using the topological phase engineering method originally suggested by Nakahara *et al.* [3], first two- and four-quantum vortices in dilute atomic BECs have been realized [4]. This method utilizes the hyperfine spin degrees of freedom of the order parameter, but finally produces a scalar condensate containing a multiquantum vortex.

Theoretical analysis has revealed that in addition to multiquantum vortices being energetically unfavorable in harmonic traps, they are also in general dynamically unstable [5,6]. This dynamical instability implies that multiquantum vortices tend to split into singly quantized ones even in the absence of dissipation, i.e., in pure condensates without a noticeable thermal gas component. The splitting dynamics is an interesting issue, because the energy and angular momentum released from the multiquantum vortex has to be redistributed in the system. The dynamics of the splitting of doubly quantized vortices created in the experiments reported in Ref. [4] was theoretically studied in Ref. [6], and two major observations were made: the initial dynamics in a cigar-shaped condensate can be modeled to some extent using only an effective two-dimensional local density analysis, and proper three-dimensional computations showed that the two vortices separating out from the doubly quantized vortex usually intertwine strongly as they split.

Recently, the splitting of doubly quantized vortices was experimentally observed and the splitting time, i.e., the time interval between the creation of the vortex and the point when two separable vortex cores were observed, was measured as a function of the peak condensate density [7]. According to Ref. [7], the temperature was low enough for one to be able to neglect the effects of thermal atoms and dissipation. The experimental results verified that the doubly quantized vortex splits into two singly quantized ones. However, since only particle densities averaged over a short slice in the longitudal direction of the condensate were measured, no intertwining of the vortices was observed.

In this Letter, we directly model the experiments reported in Ref. [7] and compare the theoretical and experimental results. We solve the full three-dimensional dynamics of the condensate using the Gross-Pitaevskii (GP) equation with time-dependent trapping potential combined with gravitational potential, closely corresponding to the experiments. From the condensate density profiles, we determine the splitting time as a function of the peak condensate density and analyze the effects of vortex intertwining. Contrary to previous theoretical results presented in Ref. [8], we find that the gravitational and the time-dependent trapping potentials together break the rotational symmetry and initiate the splitting process strongly enough to alone yield splitting times in good agreement with experiments. Thus the effect of thermal excitations is not relevant in modeling the experiments.

At low enough temperatures, the effect of the thermal gas component can be neglected and the entire gaseous

many-particle system of trapped atoms can be described by the condensate order parameter $\psi(\mathbf{r}, t)$. The dynamics of this dilute condensate is governed by the GP equation

$$i\hbar\frac{\partial}{\partial t}\psi(\mathbf{r}, t) = \left[-\frac{\hbar^2}{2m}\nabla^2 + V(\mathbf{r}, t) + g|\psi(\mathbf{r}, t)|^2\right]\psi(\mathbf{r}, t), \tag{1}$$

where $V(\mathbf{r}, t)$ is the external potential, m is the mass of the atoms, and the strength of the interactions is governed by the parameter $g = 4\pi\hbar^2 a/m$ expressed in terms of the s-wave scattering length a. The order parameter is normalized according to the total number of atoms as $\int|\psi|^2 d\mathbf{r} = N$. The stationary state solutions of the GP equation with eigenvalue μ are of the form $\psi(\mathbf{r}, t) = e^{-i\mu t/\hbar}\psi(\mathbf{r})$, and the excitation spectrum of these states can be solved from the Bogoliubov equations. The excitation spectrum determines the stability properties of the state: The existence of modes with negative energy but positive norm implies that the corresponding stationary state is energetically unstable; i.e., in the presence of dissipation and small perturbations the state will decay. Furthermore, the existence of modes with nonreal eigenfrequencies implies the state to be dynamically unstable, i.e., infinitesimal perturbations may grow exponentially in time, and thus the stationary state may decay even in the absence of dissipation.

Examples of these kind of states are the axisymmetric vortex states of the form $\psi(\mathbf{r}) = e^{i2\pi\kappa\theta}\psi(r, z)$, where (θ, r, z) denote the cylindrical coordinates and $\kappa \neq 0$. These states are energetically unstable in harmonic traps since vortices tend to spiral out of the condensate due to the buoyancy force. In addition, for quantum numbers $|\kappa| \geq 2$ there typically exists excitations with nonreal eigenfrequencies, which implies the corresponding states to be dynamically unstable and to split into singly quantized vortices. Especially, the dynamical instability of doubly quantized vortices has been demonstrated both theoretically [5,6] and experimentally [7].

In the experiments reported in Ref. [7], Shin *et al.* created a doubly quantized vortex into a dilute BEC of ^{23}Na atoms confined in a Ioffe-Pritchard trap by reversing the axial bias field B_z linearly in 12 ms. The thermal gas fraction was not discernible in the experiments, and hence dissipation was presumably extremely weak.

We have modeled these experiments by computing the time evolution of the condensate using the GP equation. Starting from energy minimized axisymmetric doubly quantized vortex states corresponding to different values of the density parameter $an_z = a\int|\psi(x, y, 0)|^2 dxdy$ (at the center of the condensate $z = 0$), we calculate the condensate dynamics in the time-dependent external potential corresponding to the Ioffe-Pritchard trap and gravity. Taking the gravity into account turns out to be crucial, since only the combined potential breaks the rotational symmetry about the vortex axis and is shown to initiate the vortex splitting process. In the computations, we used finite-difference discretization. The initial states were found using imaginary time integration, and the time evolution was calculated by a split operator method.

In a Ioffe-Pritchard trap, the square of the total magnetic field strength to the second order in the radial and axial coordinates is given by [9]

$$B^2(t) = C^2 r^2 + B_z(t)A[z^2 - \tfrac{1}{2}r^2] + B_z^2(t), \tag{2}$$

where A determines the curvature of the initial axial field, C characterizes the strength of the field generated by the Ioffe bars, and B_z is the axial bias field. Including gravity, the external potential for the weak-field seeking state of a spin-1 condensate reads

$$V(\mathbf{r}, t) = -g_L\mu_B B(t) + Gmx, \tag{3}$$

where $g_L = -1/4$ and G is the gravitational acceleration. At $t = 0$, the term $B_z(0)$ dominates in Eq. (2) in the vicinity of the center of the trap, and hence the total potential is approximately harmonic in the condensate region. The initial potential is of the form

$$V(\mathbf{r}, 0) \approx \tfrac{1}{2}m\omega_z^2 z^2 + \tfrac{1}{2}m\omega_r^2\{y^2 + [x - x_0(0)]^2\} + \text{const.} \tag{4}$$

According to the experiments, we choose the parameters A and C such that the axial and radial trapping frequencies are $\omega_z = 12$ Hz and $\omega_r = 220$ Hz, respectively. In addition, $x_0(t) = Gm|B_z(t)|/(g_L\mu_B[C^2 - B_z(t)A/2])$ is the location of the minimum of $V(\mathbf{r}, t)$. The bias field B_z vanishes at $t = 6$ ms, and hence the radial trapping potential is linear with respect to r. As $B_z(t)$ passes through zero, the sign of A is changed in order to keep the axial field confining. After the reversal of B_z at $t = 12$ ms, the potential assumes its initial form.

The main results of our simulations are presented in Figs. 1–5. Figure 1 shows the location $x_0(t)$ of the minimum of $V(\mathbf{r}, t)$ and the computed location of the center of

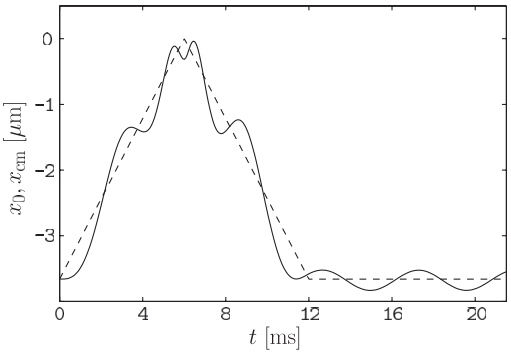

FIG. 1. Location $x_0(t)$ of the minimum of the total potential $V(\mathbf{r}, t)$ (dashed line) and the location $x_{\text{c.m.}}(t)$ of the center of mass of the condensate (solid line). Because of inertia, the condensate lags behind the center of the potential, which breaks the rotational symmetry of the system. This perturbation initiates the splitting of the doubly quantized vortex.

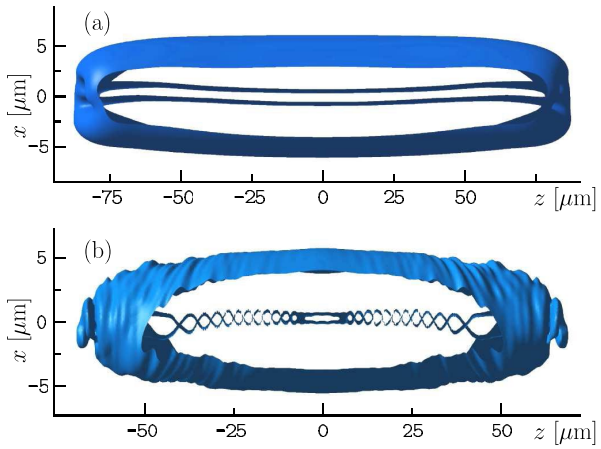

FIG. 2 (color online). Isosurfaces of condensate densities for (a) $an_z = 2.56$ at time $t = 10.9$ ms and (b) $an_z = 14.4$ at $t = 39.8$ ms. The values defining the isosurfaces are of the order of 1% of the maximal density.

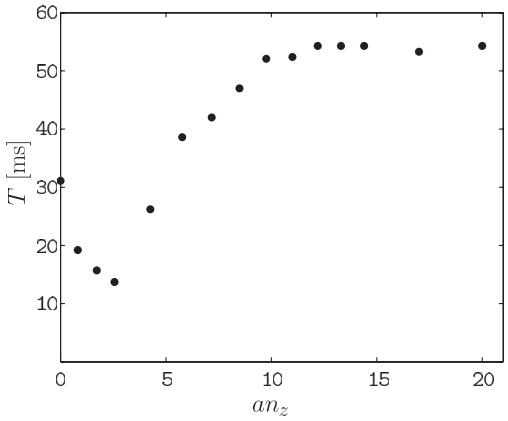

FIG. 4. The splitting time T of a doubly quantized vortex as a function of an_z. The splitting time decreases rapidly as interactions are introduced in the system attaining its minimum value around $an_z \approx 3$. As the interaction strength is increased further, T increases due to the stable region near the center of the condensate. The splitting time saturates roughly to $T \approx 55$ ms for interaction strengths $an_z > 10$.

mass of the condensate as functions of time. Since the condensate slightly lags behind the time-dependent center of the potential, the rotational symmetry of the initial condensate state is broken. This perturbation also excites the dipole mode of the condensate, which is manifested by the oscillatory behavior of the center of mass after the potential has returned to its initial form. However, the amplitude of the dipole oscillation is less than 3% of the diameter of the condensate.

In the effectively two-dimensional case, for which $\omega_z = 0$ and the condensate is homogenous in the z direction, the imaginary part of the frequency corresponding to the eigenmode responsible for the dynamical instability assumes a quasiperiodic form as function of an_z [5,6], and vanishes in some regions of the parameter an_z. Also in three-dimensional cigar-shaped condensates, regions where splitting is significantly faster or slower are clearly distinguishable. In Fig. 2(a), the isosurface of a condensate with $an_z = 2.56$ is plotted at $t = 10.9$ ms. In this case, the splitting process is initiated along the whole length of the condensate, to be contrasted to the case $an_z = 14.4$ at $t = 39.8$ ms depicted in Fig. 2(b), in which the splitting initiates only from the ends and center of the condensate. Since the precession frequency of the vortices depends on their distance, intertwining is observed in cases which contain such stable regions. Note also the distinctive surface mode excitations due to the additional energy and angular momentum released in the splitting process.

We determine the splitting times T from the density profiles $\bar{n}(x, y) = \int_{-15\ \mu\mathrm{m}}^{15\ \mu\mathrm{m}} |\psi(x, y, z)|^2 dz$, mimicking the tomographic imaging technique used in the experiments [7]. Roughly, when two minima are observed in the density profile with their separation exceeding the diameter of a singly quantized vortex, the doubly quantized vortex was considered split in the experiments. We define the diameter of a single quantum vortex as twice the distance from the vortex center to 75% of the maximum value in the density profile $\bar{n}(x, y)$ of the stationary single quantum vortex state. In Fig. 3, the density profile $\bar{n}(x, y)$ is plotted at $t = 10.9$, 14.5, and 18.1 ms for $an_z = 2.56$. From the isosurface plot in Fig. 2(a), one might argue that the vortex has already split at $t = 10.9$ ms, but from the density plot in Fig. 3(a)

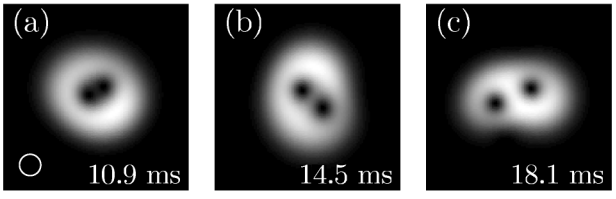

FIG. 3. Density profiles $\bar{n}(x, y) = \int_{-15\ \mu\mathrm{m}}^{15\ \mu\mathrm{m}} |\psi(x, y, z)|^2 dz$ of a condensate with $an_z = 2.56$ at three different times t. The splitting time of the doubly quantized vortex is determined by comparing the separation of the local minima in the density profiles to the diameter of a single quantum vortex represented by the white circle corresponding to a diameter of 1.8 μm in (a). The field of view in these figures is 13.5 μm $\times$ 13.5 μm.

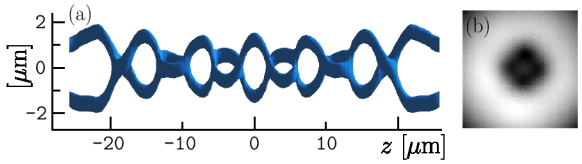

FIG. 5 (color online). (a) Isosurface plot of the vortex chain structure formed in a condensate with $an_z = 11.0$ at $t = 54.3$ ms. The value defining the isosurface is of the order of 10% of the maximal density. (b) The corresponding density profile $\bar{n}(x, y)$ shows four minima instead of two. The field of view in (b) is 6.8 μm $\times$ 6.8 μm.

no such conclusions can be made. In Fig. 3(b), the distance of the local density minima already exceeds the single quantum vortex diameter and a detailed analysis yields $T = 13.7$ ms for the splitting time.

Figure 4 shows the splitting times for values of an_z ranging from 0 to 20 corresponding up to $N = 8 \times 10^5$ ^{23}Na atoms. A finite splitting time T is obtained also in the noninteracting case, which concurs with the observation that persistent stationary currents exist in the noninteracting case only if the spectrum of the Hamiltonian is degenerate [10]. For $an_z < 3$, T decreases with increasing interaction strength. This effect is also visible in the recent theoretical study of Ref. [8], but it is not clearly observable in the experimental results [7]. The splitting time attains its minimum at $an_z \approx 3$, which is in good agreement with the Bogoliubov eigenvalue spectrum analysis accomplished in Refs. [5,6]. For $an_z > 10$, T saturates to a finite value, which is in fair agreement with the experimental results [7], although the saturation value is roughly 20% larger in the experiments. The overall form of the computed splitting times as a function of an_z agrees well with the experimental data, and the agreement is surprisingly good even for the quantitative results. Note that the computed splitting times are generally somewhat smaller than the measured ones, implying that the time dependency of the external potential alone gives sufficiently strong impetus for the splitting, and the additional contribution of thermal fluctuations is not needed. We have also checked that the splitting is not initiated by numerical instabilities: when gravity is neglected, there is no sign of splitting or rotational symmetry breaking even after 50 ms for $an_z \approx 4$.

A few points should be taken into account when comparing the computed splitting times to experimental values. First, our simulations neglect the multicomponent nature of the condensate when the multiquantum vortex is created by reversing the bias field B_z. In our simulations, we begin with the multiquantum vortex state already at $t = 0$. Effectively, this difference can roughly be assumed to shorten the computed lifetimes of the multiquantum vortices by half of the reversing time, i.e., by 6 ms. However, this difference is presumably more than compensated by the fact that in the simulations we start counting the splitting time from the beginning of the perturbation, whereas in the experiments the time is counted from the end of the perturbation. We have also checked computationally that the presence of the vortex already at $t = 0$ only slightly alters the amplitude of the quadrupole oscillations induced by the external potential in the beginning of the bias field reversal—these oscillations finally initiate the vortex splitting process. Second, in the experiments the condensate is let to expand freely for 15 ms before imaging, and in our results possible changes in the relative vortex separation during this expansion period are neglected.

For values near $an_z \approx 10$, where there is a stable region at the center of the condensate, a peculiar linked chain-type structure for the intertwined vortices was observed; see Fig. 5(a). The z-integrated density profile observed with the tomographic imaging technique is shown in Fig. 5(b). In this case, there are four distinct minima in the density profile instead of the usual two. This effect could explain the observation interpreted as crossing of the vortex lines in the experiments.

In conclusion, we have calculated the splitting times of doubly quantized vortices as a function of peak particle density by solving the time-dependent Gross-Pitaevskii equation numerically, mimicking closely the experiments reported in Ref. [7]. The results are in good agreement with the experimental data, confirming that thermal fluctuations are not required to explain the measured splitting times. Instead, the main impetus for the rotational symmetry breaking and splitting is the combination of the gravitational and time-dependent trapping potentials.

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