# Nonexponential Decay Via Tunneling in Tight-Binding Lattices and the Optical Zeno Effect

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An exactly-solvable model for the decay of a metastable state coupled to a semi-infinite, tight-binding lattice, showing large deviations from exponential decay in the strong coupling regime, is presented. An optical realization of the lattice model, based on discrete diffraction in a semi-infinite array of tunneling-coupled optical waveguides, is proposed to test nonexponential decay and for the observation of an optical analog of the quantum Zeno effect.

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The understanding and control of the decay process of an unstable quantum state has long been a subject of debate in different areas of physics. Though an exponential law is known to be a good phenomenological fit to many decay phenomena, quantum mechanics ensures that the survival probability $P(t)$ is definitely *not* exponential at short and long times (see, e.g., [1–3]). In particular, at short times $P(t)$ always shows a parabolic decay, i.e. $dP/dt \to 0$ as $t \to 0$. These universal features have been extensively investigated in some specific models describing the tunneling escape of a particle through a potential barrier [1,4,5], or in the framework of the exactly-solvable Friedrichs-Lee Hamiltonian [6–10], which describes the decay of a discrete state coupled to a continuum. The short-time features of the decay process have attracted much attention because they can lead, under certain conditions, to either the deceleration (Zeno effect) or the acceleration (anti-Zeno effect) of the decay by frequent observations of the system (see, e.g., [8,9,11] and references therein). Evidences of nonexponential decay features at short times and the observation of the related Zeno and anti-Zeno effects have been reported in recent experiments on quantum tunneling of trapped sodium atoms in accelerating optical lattices [12]. Similar effects have been proposed to occur for quantum tunneling in analogous macroscopic systems, such as Josephson junctions [13].

In this Letter, a novel and exactly-solvable model of nonexponential decay of an unstable state tunneling coupled to a tight-binding lattice is presented. A simple and experimentally accessible realization of the model, based on discrete diffraction of photons in an array of optical waveguides [14], is proposed along with an optical analog of the quantum Zeno effect. To set our model in a general context, we consider a semi-infinite lattice described by the tight-binding Hamiltonian [Fig. 1(a)]:

$$H_{\mathrm{TB}} = -\hbar \sum_{n=1}^{\infty} \Delta_n (|n\rangle\langle n+1| + |n+1\rangle\langle n|), \qquad (1)$$

where $|n\rangle$ ($n \geq 1$) is the state localized at the $n$-th site of the lattice and $\Delta_n$ is the hopping amplitude between adjacent sites $|n\rangle$ and $|n+1\rangle$. We assume that for $n \geq 2$, the lattice is periodic so that, after a rescaling of time $t$, we may assume $\Delta_n = 1$ for $n \geq 2$. The boundary site $|1\rangle$ is then coupled to the periodic lattice by a hopping amplitude $\Delta_1 = \Delta$, which is assumed to be smaller than 1. The tight-binding Hamiltonian (1) has been often used as a simple model to describe coherent transport properties and tunneling phenomena in different physical systems, including semiconductor superlattices [15], arrays of coupled quantum dots [16], Bose-Einstein condensates in optical lattices [17], and arrays of optical waveguides [14,18]. In particular, model (1) can be derived from the continuous Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + V(x)\psi, \qquad (2)$$

with a potential $V(x) = \sum_{n=1}^{\infty} V_w(x - x_n)$ describing a semi-infinite chain of identical symmetric quantum wells $V_w(x)$ [$V_w(-x) = V_w(x)$ and $V_w(x) \to 0$ for $x \to \infty$], placed at distances $x_{n+1} - x_n = a$ for $n \geq 2$ and $x_2 - x_1 = a_0 > a$ [see Fig. 1(a)]. If the individual potential well $V_w(x)$ supports a single bounded mode $\varphi(x)$ of energy

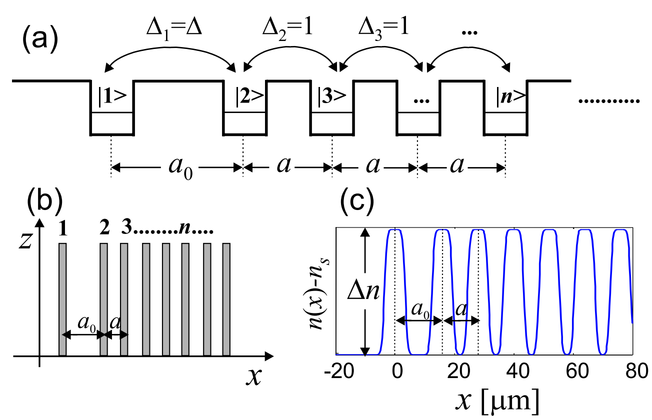


FIG. 1 (color online). (a) The semi-infinite tight-binding lattice model. (b) Optical realization of the tight-binding model based on an array of coupled optical waveguides. (c) Refractive index profile $n(x) - n_s$ of the waveguide array used in the numerical simulations (parameter values are $n_s = 2.138$, $\lambda = 1.55$ $\mu$m, $\Delta n = 2.4 \times 10^{-3}$, and $a = 12$ $\mu$m).

 

$E$ and if tunneling-induced coupling of adjacent wells is weak, Eq. (2) can be reduced to the discrete model (1) by means of a tight-binding [17,19] or a variational [20] analysis. After expanding the state $|\psi\rangle$ of the system as $|\psi\rangle = \sum_n c_n(t)\exp(-iEt/\hbar)|n\rangle$, where $|n\rangle = \varphi(x - x_n)$ is the localized state at the $n$-th well in the chain, in the nearest-neighbor approximation from Eq. (2), one can derive the following equations of motion for $c_n$:

$$i\dot{c}_1 = -\Delta c_2, \qquad i\dot{c}_2 = -c_3 - \Delta c_1, \\ i\dot{c}_n = -(c_{n+1} + c_{n-1}) \quad \text{for } n \geq 3, \tag{3}$$

where $\Delta \simeq [\int dx\varphi(x - a_0)V_w(x)\varphi(x)]/[\int dx\varphi(x - a)V_w(x)\varphi(x)]$ is the normalized hopping amplitude between states $|1\rangle$ and $|2\rangle$. For $\Delta = 0$, i.e. for $a_0/a \to \infty$, the site $|1\rangle$ is decoupled from the other lattice sites, and if the system is initially prepared in state $|1\rangle$, it does not decay; as $\Delta$ is increased, tunneling escape is allowed, and state $|1\rangle$ becomes metastable. The limits $\Delta \to 0$ and $\Delta \to 1$ correspond to the weak and strong coupling regimes, respectively. The occupation probability of site $|1\rangle$ at time $t$ is given by $P(t) = |c_1(t)|^2$. Following Gamow's approach to quantum tunneling decay [4], the "natural" decay rate $\gamma_0$ of state $|1\rangle$, which would correspond to an exponential decay law $P(t) = \exp(-\gamma_0 t)$, can be readily calculated by looking for complex energy eigenfunctions of $H_{\mathrm{TB}}$ with outgoing boundary conditions (Gamow's states), yielding

$$\gamma_0 = 2\Delta^2(1 - \Delta^2)^{-1/2}. \tag{4}$$

However, the exponential decay law turns out to be incorrect, especially in the strong coupling regime $\Delta \to 1$ where it fails to reproduce the exact decay law at *any* time scale. According to Ref. [9], one can introduce an *effective* decay rate $\gamma_{\mathrm{eff}}(t)$ by the relation $\gamma_{\mathrm{eff}}(t) = -(1/t)\ln|c_1(t)|^2$, so that any deviation of $\gamma_{\mathrm{eff}}(t)$ from $\gamma_0$ is a signature of nonexponential decay. In addition, the eventual intersection $\gamma_{\mathrm{eff}}(t) = \gamma_0$ rules the transition from Zeno to anti-Zeno effects for repetitive measurements [9]. In order to determine the *exact* law for the survival probability $P(t)$, one has to calculate the eigenfunctions of (1) and construct a suitable superposition of them corresponding, at $t = 0$, to a particle localized in the well $|1\rangle$, i.e. to $c_n(0) = \delta_{n,1}$. The tight-binding Hamiltonian (1) has a continuous spectrum of eigenfunctions [21] which can be calculated by separation of variables and correspond to $c_n(t) = u_n(Q) \times \exp[i\Omega(Q)t]$, where $\Omega(Q) = 2\cos Q$ is the dispersion curve of the tight-binding lattice band, $-\pi < Q < \pi$ varies in the first Brillouin zone, and

$$u_1 = \Delta(1 + r)/(2\cos Q), \\ u_n = \exp[-iQ(n-2)] + r\exp[iQ(n-2)] \quad (n \geq 2). \tag{5}$$

In Eq. (5), $r = r(Q)$ is the reflection coefficient for Bloch waves at the boundary of the semi-infinite lattice and reads explicitly

$$r(Q) = -\frac{\Delta^2 - 2\cos Q\exp(iQ)}{\Delta^2 - 2\cos Q\exp(-iQ)}. \tag{6}$$

To study the decay process, we construct a superposition of the eigenstates, $c_n(t) = \int_{-\pi}^{\pi} dQ F(Q) u_n(Q)\exp[i\Omega(Q)t]$, where the spectrum $F(Q)$ is determined by the initial conditions $c_n(0) = \delta_{n,1}$. Using an iterative procedure that will be described in detail elsewhere, one can show that the searched spectrum is given by $F(Q) = -(2\pi\Delta)^{-1} \times [\Delta^2\exp(iQ) - 2\cos Q]/[\Delta^2 - 1 - \exp(2iQ)]$. Therefore the *exact* decay law for the occupation amplitude of site $|1\rangle$ is given by

$$c_1(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} dQ\exp(2it\cos Q)\frac{1 - \exp(-2iQ)}{1 + \alpha^2\exp(-2iQ)}, \tag{7}$$

where $\alpha \equiv (1 - \Delta^2)^{1/2}$. The short-time decay of $|c_1(t)|^2$ is obviously parabolic; the long-time behavior of $c_1(t)$ can be calculated by use of the method of the stationary phase, yielding the oscillatory power-law decay

$$c_1(t) \sim \frac{1}{\sqrt{\pi}}\frac{1 - \alpha^2}{(1 + \alpha^2)^2}\frac{1}{t^{3/2}}\cos(2t - 3\pi/4) \quad \text{as } t \to \infty. \tag{8}$$

In order to extract the exponential decay part from $c_1(t)$, after setting $z = \exp(iQ)$, it is worth rewriting Eq. (7) as an integral in the complex plane

$$c_1(t) = \frac{1}{2\pi i}\oint_\sigma dz\exp\left[it\left(z + \frac{1}{z}\right)\right]\frac{z^2 - 1}{z(z^2 + \alpha^2)}, \tag{9}$$

where the contour $\sigma$ is the unit circle $|z| = 1$. The integral (9) can be evaluated by use of the residue theorem. Note that for $\Delta = 1$, there is only one singularity at $z = 0$, and from residue theorem one obtains

$$c_1(t) = (1/t)J_1(2t), \tag{10}$$

which shows that, in the strong coupling regime, the decay greatly deviates from an exponential law at *any* time scale. For $\Delta < 1$, there are three singularities, at $z = 0$ and $z = \pm i\alpha$, inside the contour $\sigma$. The residue associated with the singularity $z = -i\alpha$ yields an exponentially-decaying term, whereas the sum of residues at $z = 0$ and $z = i\alpha$ yields a bounded function $s(t)$, which can be written as a Neumann series. Precisely, one can write

$$c_1(t) = \sqrt{Z}\exp(-\gamma_0 t/2) + s(t), \tag{11}$$

where $\gamma_0$ is the natural decay rate as given by Gamow's theory [Eq. (4)], $\sqrt{Z} \equiv (\alpha^2 + 1)/(2\alpha^2)$, and

$$s(t) = J_0(2t) + \left(1 + \frac{1}{\alpha^2}\right)\left[\frac{1}{2}\sum_{l=-\infty}^{\infty}\frac{J_l(2t)}{\alpha^l} - \sum_{l=0}^{\infty}\frac{J_{2l}(2t)}{\alpha^{2l}}\right] \tag{12}$$

is the correction to the exponential decay term. The decomposition (11) is meaningful in the weak coupling regime ($\Delta \to 0$) since, in this limit, one can show that the contribution $s(t)$ is small and of order $\sim\Delta^2$. The appearance of nonexponential features in the decay dynamics when approaching the strong coupling limit is clearly shown in Fig. 2, where the numerically-computed behavior of the effective decay rate $\gamma_{\rm eff}(t)$ is shown for a few values of $\Delta$, together with the temporal evolution of amplitudes $|c_n(t)|$. The appearance of strong oscillations in the $\gamma_{\rm eff}(t)$ curve when the coupling strength increases is a clear signature of an oscillatory decay dynamics which sets in *even* at intermediate time scales. Consider now the case of projective measurements of state $|1\rangle$ at time intervals $t = \tau$. In the weak coupling limit, where the decay deviates from an exponential law solely at short and long times, deceleration of the decay (Zeno effect) occurs for $\tau < \tau^*$, where $\tau^*$ is the smallest root of the equation $\gamma_{\rm eff}(\tau^*) = \gamma_0$ [9]; for instance, for parameter values of Fig. 2(a), one has $\tau^* \sim 85$. In the strong coupling regime [Fig. 2(c)], the decay is highly oscillatory, and acceleration of the decay (anti-Zeno effect) may be observed for a value of $\tau$ close to, for example, the first peak of $\gamma_{\rm eff}$, where $\gamma_{\rm eff}(\tau)$ is larger than $\gamma_0$; for instance, for parameter values of Fig. 2(c), anti-Zeno effect may be observed for $\tau \sim 2.34$. In this case, repetitive observations correspond to suppression of the oscillatory tails in the decay process.

Physical realizations of the tight-binding model (1) are provided by electron transport in a chain of tunneling-coupled semiconductor quantum wells [15] or by discrete diffraction of photons in a semi-infinite array of tunneling-coupled optical waveguides, where the temporal variable $t$ of the quantum problem is mapped into the spatial propagation coordinate $z$ along the array [Fig. 1(b)]. Here we consider in detail the latter optical system since it shows several advantages: (i) Visualization of the tunneling dynamics is experimentally accessible [18,22], and a quantitative measure of light decay can be done by, for example, Near-Field Scanning Optical Microscopy (NSOM) techniques [23]; (ii) Preparation of the system on state $|1\rangle$ is simply realized by initial excitation of the boundary waveguide by a focused laser beam; (iii) Light diffraction experiments in waveguide arrays have successfully confirmed the reliability of the tight-binding model [14,18]; (iv) Transport of photons instead of charged particles (e.g. electrons) avoids the occurrence of dephasing or many-body effects, making waveguide-based optical structures an ideal laboratory for the observation of several analogs of coherent quantum dynamical effects (see, e.g. [22]). Beautiful optical analogs of Bloch oscillations [14,18,22], Landau-Zener tunneling [22], adiabatic stabilization of atoms in strong fields [24], and coherent control of quantum tunneling [25] have been indeed reported in recent optical experiments.

Light propagation in the waveguide array is described by Eq. (2) in which the temporal variable $t$ is replaced by the spatial propagation coordinate $z$, $\hbar = \lambda/(2\pi)$ is the reduced wavelength of photons, $m = n_s$ is the refractive index of the array substrate, $V(x) \simeq n_s - n(x)$, and $n(x)$ is the array refractive index profile (see, e.g., [24,25]). As an example, Fig. 3 shows the discrete diffraction patterns and corresponding behavior of light trapped in waveguide $|1\rangle$ as obtained by a numerical analysis of Eq. (2) using a standard beam propagation method with absorbing boundary conditions [26]; initial condition corresponds to excitation of waveguide $|1\rangle$ in its fundamental mode, i.e.

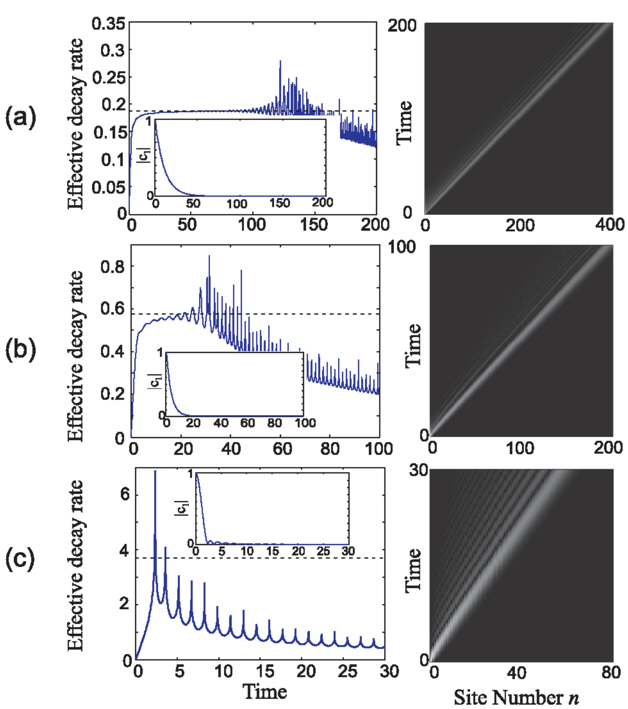


FIG. 2 (color online). Left: Behavior of the effective decay rate $\gamma_{\rm eff}$ and amplitude $|c_1(t)|$ (insets) versus time. Right: Grey scale image of $|c_n(t)|$. In (a), $\Delta = 0.3$; in (b), $\Delta = 0.5$; in (c), $\Delta = 0.9$. The horizontal dashed lines are the natural decay rate $\gamma_0$.

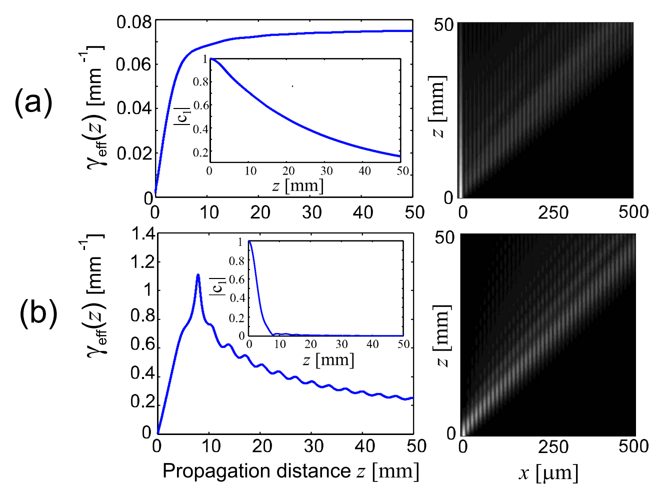


FIG. 3 (color online). Tunneling decay dynamics in a $L = 50$ mm-long, semi-infinite waveguide array (left column) and corresponding discrete diffraction patterns (right column). (a) Weak coupling regime [$a = 12$ $\mu$m and $a_0 = 16$ $\mu$m, corresponding to $\Delta \sim 0.28$]; (b) strong coupling regime [$a = 12$ $\mu$m and $a_0 = 12.5$ $\mu$m, corresponding to $\Delta \sim 0.86$].

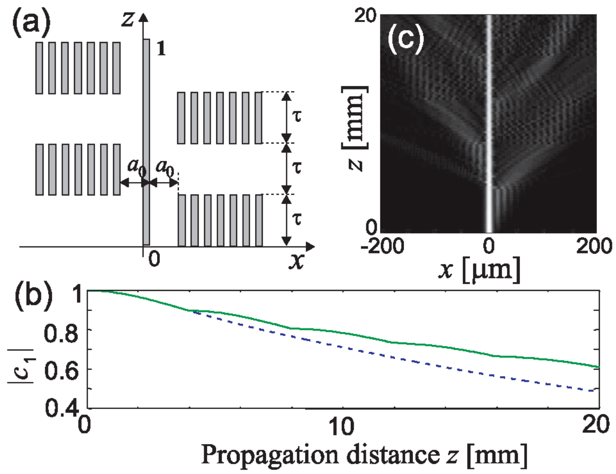


FIG. 4 (color online). (a) Schematic of a waveguide array for the observation of the optical Zeno effect. (b) Numerically-computed behavior of mode amplitude $|c_1|$ trapped in waveguide $|1\rangle$ (solid curved line) versus propagation distance in a $L = 20$ mm-long array for $\tau = 4$ mm, $a_0 = 16$ $\mu$m, and $a = 12$ $\mu$m. The dashed curved line is the behavior corresponding to Fig. 3(a). (c) Grey scale discrete diffraction pattern along the array.

$\psi(x, 0) = \varphi(x)$. The refractive index profile of the semi-infinite array used in the simulations is plotted in Fig. 1(c) for parameter values which typically apply to lithium-niobate waveguides [24]. Note that, as $\Delta$ is increased, nonexponential features are clearly visible. However, as compared to the tight-binding results, the peaked structure of $\gamma_{\mathrm{eff}}(t)$ obtained from the continuous model (2) is smoothed [compare, e.g. Fig. 2(c) and 3(b)]. In order to reproduce the optical analog of the quantum Zeno effect in the waveguide system, one can adopt the array configuration shown in Fig. 4(a), in which a straight waveguide $|1\rangle$ is periodically coupled, at equally-spaced distances $z = \tau$, to semi-infinite arrays of finite length $\tau$ placed on alternating sides of the waveguide. At each section where the lateral arrays end, light trapped in the interrupted waveguides is scattered out, and solely a negligible fraction of it will be recoupled into the waveguides at the next section of the array. Therefore, at planes $z = \tau, 2\tau, 3\tau, \ldots$ one can assume, at first approximation, that a collapse of the state $\psi(x, z)$ into the fundamental mode $\varphi(x)$ of waveguide $|1\rangle$ occurs, thus simulating the "wavepacket collapse" of an ideal quantum measurement. An example of deceleration of the decay via tunneling in the alternating array, analogous to the quantum Zeno effect, is shown in Figs. 4(b) and 4(c).

In conclusion, an exactly-solvable model for the tunneling escape dynamics of a metastable state coupled to a tight-binding lattice has been presented, and its optical realization—including an optical analog of the quantum Zeno effect—has been proposed in an array of tunneling-coupled optical waveguides.

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