Distribution of Anomalous Exponents of Natural Images

G. Frenkel,[1] E. Katzav,[2] M. Schwartz,[3] and N. Sochen[1]

[1]School of Mathematics, Raymond and Beverly Sackler Faculty of Exact Sciences, Tel Aviv University, Ramat Aviv 69978, Israel
[2]Department of Chemical Physics, The Weizmann Institute of Science, Rehovot 76100, Israel
[3]School of Physics and Astronomy, Raymond and Beverly Sackler Faculty of Exact Sciences, Tel Aviv University, Ramat Aviv 69978, Israel

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Recent studies of correlations of intensity in databases of natural images revealed a remarkable property. The two point correlations are described in terms of power law behavior, with an exponent which seems to be robust. In the present Letter we consider the statistical meaning of that result. We study many individual images of one of the databases considered. We find that the same law characterizing the correlations in the whole database governs also images randomly chosen from that database, with one essential difference. The exponent characterizing each image is specific and differs from the exponent characterizing the whole database. The distribution of single image exponents has been measured and found to exhibit a rather heavy tail. The database exponent cannot, thus, be considered as a statistical representative of a single image exponent. Possible reasons for the diversity in image exponents are discussed.

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Much attention has been recently focused on the statistical properties of natural images [1–4]. There are a number of practical reasons that motivate such studies. Images almost always represent a 2D projection of a 3D environment. This means that the spatial adjacency relations are changed [5]. In addition, natural images represent a very diverse clutter of objects. Thus the statistical properties of images are closely related to physical and biological systems where large diversity and strong correlation between distant objects are present. Another motivation is that statistical properties of various features in the image enter as a prior in various image processing and computer vision tasks. The prior is given as a density function (a probability distribution function) that characterizes the probability of a given image to be a "real" natural image. Such a characterization is the key to success in various image processing and computer vision tasks, such as transmission, compression, and denoising to name just a few. These are related to the information content of the image. Yet, perhaps more remarkable are the observations of correlations of the light intensity that do not seem to be related to information content in the image. Similar correlations exist in ferromagnets at their critical point or in systems of self-organized criticality. It has been shown by several authors [1–4] that intensity correlations exhibit scaling and even more remarkable that scaling showed promise of being universal. Ruderman and Bialek [1] have considered a set of images where each image is considered as an array of pixels in which the light intensity is recorded. After replacing the value of the intensity at each pixel, i, by its logarithm, $\phi(i)$, the power spectrum of the whole set of images is taken. The power spectrum is found to be proportional to $q^{-(2-\eta)}$, where q is the absolute value of the "momentum" vector $\mathbf{q}$, corresponding to the appropriate Fourier components of the image. Such a behavior is typical to the Fourier transform of the spin-spin correlation function at the point of phase transition. In that case the exponent η is known as the anomalous dimension, is positive, and is usually not large [6–8]. Such power law behavior is also characteristic of a system exhibiting self-organized criticality, such as the Kardar-Parisi-Zhang (KPZ) system [9–13] in which the relevant real space correlation corresponds to the typical height variation of a deposited surface over a distance r. The power spectrum corresponds to the correlation $\langle h_{\mathbf{q}} h_{-\mathbf{q}} \rangle$, where $h_{\mathbf{q}}$ is the Fourier transform of the height function defining the two dimensional surface. In that case the exponent η is negative, corresponding to the fact that the surface is self-affine. Grey-value images can be viewed, of course, as a surface in a three-dimensional space consisting of two spatial dimensions and one feature dimension, the intensity. This observation has been recently generalized to represent images with more features, such as color, as the embedding of surfaces in a spatial-feature higher dimensional spaces [14,15]. A recent review article [4] gives a comprehensive discussion of the phenomenon of scale invariance in natural images, starting from old observations of television engineers [16,17] to more recent work on natural images starting in the late 1980s and continuing ever since [1,3,4,18,19]. We will concentrate our discussion, however, around the studies of two groups of researchers, because those studies lead, in our mind, to results that are most puzzling.

Ruderman and Bialek [1] find $\eta = 0.19$. As pointed out by Ruderman [2], the scaling form of the power spectrum determines the scaling form of the two point difference function,

$$D(\mathbf{r}) = \langle [\phi(\mathbf{r}) - \phi(0)]^2 \rangle, \qquad (1)$$

to be given by

$$D(\mathbf{r}) = D_1 - D_2 r^{-\eta}, \quad (2)$$

where D_1 and D_2 are constants. Indeed direct measurement of the difference function obtained from averaging over a small database of images taken from wood scenery yields also $\eta = 0.19$.

The work described above, as well as the work of other researchers that found scaling in natural images [20], is based on the study of databases that are limited to images of a particular type. More surprising, however, is the finding of Huang and Mumford [3]. They studied these correlations over a large archive of 4000 images, provided by van Hateren and van der Schaaf [21]. Taking pairs of points from all images together, they were able to show that the form given by Eq. (2) above describes the behavior of the difference function over a range of distances between 4 and 32 pixels with the same exponent, $\eta = 0.19$. The archive they used, however, is very diverse and contains scenes of forest, buildings, grass, clouds, roads, and rivers all in the same image and images taken from very different angles. The fact that the anomalous dimension observed fits that obtained in Refs. [1,2], in spite of the obvious diversity of the different images in the archive, is remarkable and puzzling indeed. Our experience in the field of continuous phase transitions and of self-organized criticality is that various exponents characterizing the system are universal. Namely, those exponents are exactly the same for systems which are quite different from each other as long as they share a small number of some physical attributes such as dimension, symmetry, and range of interaction. The puzzlement with the results described above is that they indicate that natural images are universal in some sense. This is extremely surprising and yet very important if true.

The purpose of the present Letter is mainly to understand whether the robustness of the exponent really indicates universality. This is done in two steps. First, we try to reproduce the result of Huang and Mumford [3] and then to study its statistical significance, namely, to ask whether the database exponent is representative of single images in that database. This is achieved by showing that each image separately can indeed be described by a difference function of the form given above (2) and constructing the distribution function of the separate η's. Our purpose here is not to explain the origin of the form of the two point difference function in each image or in the database but to show that they exist and to study the statistical meaning of our finding.

We use the archive of van Hateren and van der Schaaf [21] that was used by Huang and Mumford [3]. For each image, given as a 1024×1536 array of intensities, we calculate the difference function $D(\mathbf{r})$ in the following way. We choose at random 5×10^6 pairs of points in the image that are separated by distances between 0 and 32 pixels and are oriented in different directions. (The reason for our choosing distances only between 0 and 32 pixels is our intention to repeat exactly the procedure of Huang and Mumford [3]. It is clear that consideration of larger distances may be of importance, but this is postponed to future publications.) The difference function of each image is obtained by averaging $[\phi(\mathbf{r}) - \phi(\mathbf{0})]^2$ for each distance in the range of up to 32 pixels over the pairs corresponding to that distance. The database difference function is the average over images of the image difference function. We obtain the database difference function by averaging over the first 1500 images chosen from the archive. We then fit the data by the form given in Eq. (2). The observed results are depicted against the best fit in Fig. 1. The pluses represent the observed results, while the continuous line represents the best fit. The value of η corresponding to the best fit is $\eta = 0.19$. This value varies a little due to averaging over different numbers of images. In our measurements the variation was of the order of $\Delta\eta = \pm 0.01$ (for instance, for the first 1100 images of the archive the value $\eta = 0.18$ is obtained). The result of Huang and Mumford is thus reproduced.

Our next question is whether the image difference functions can be fitted by the same form of behavior as the database difference function but not necessarily with the same constants. We do expect, however, that the observed image difference function will be noisier than the database difference function just because the average is on a number of pairs (at each distance) that is by the order of 1000 smaller. We have obtained the fit for all our images, but naturally we can present it only for a small number as depicted in Figs. 2 and 3.

The obtained fit is good. It is characteristic of the images with a relatively large exponent to fluctuate a bit around the fitted line (Fig. 3, right panel). Note that we perform here a nonlinear fit that is not necessarily convex and may suffer from local minima. Since the exponents that describe the individual images are not of a definite sign, care should be taken when fitting the data. Search procedures starting, say, from a positive exponent, in a case where the exponent is

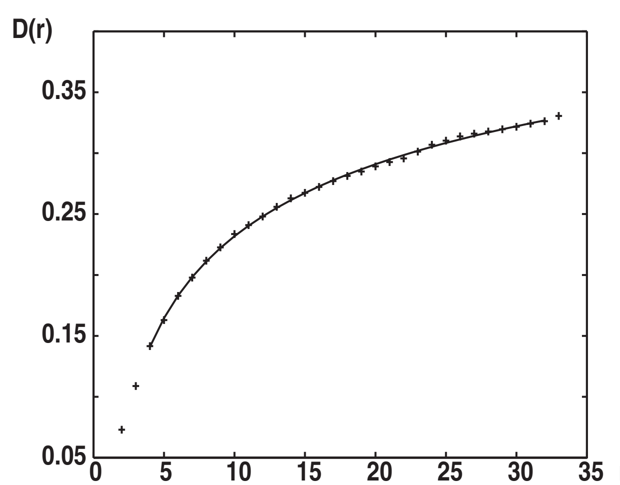

FIG. 1. Averaged difference function. The average is taken over 1500 images, $\eta = 0.19$.

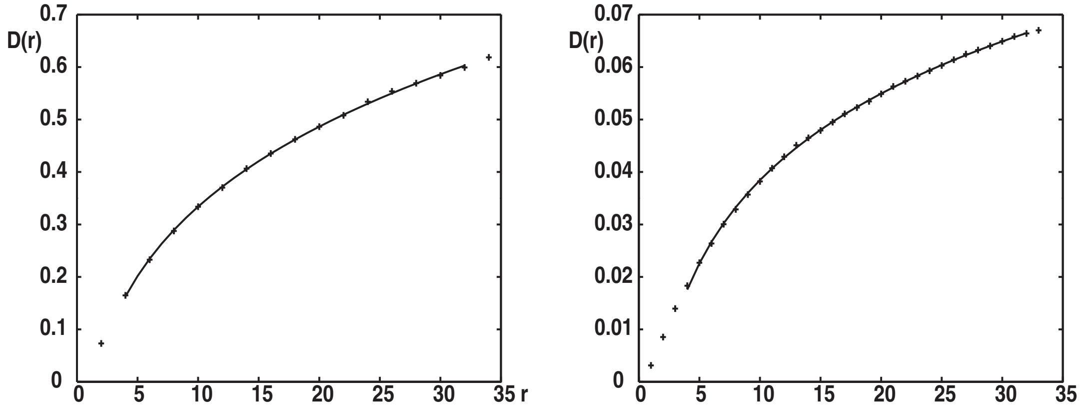

FIG. 2. Left panel: Image 1401, $\eta = -0.2$. Right panel: Image 262, $\eta = -0.05$.

actually negative, may end in a small positive exponent and in a fit which is less than satisfactory. We obviously see that the value of anomalous dimension differs from image to image. Note that the results we give for the anomalous dimensions for different images is based on measurements in the range of 0–32 pixels in accordance with the measurements of Huang and Mumford on the full database. Now, if the value of the exponent was the same for all individual images, the value of the database anomalous dimension would have been identical to the value of the images. For the database value to be considered as a representative of the values on the images, the distribution of the image anomalous dimension should be narrow. Figure 4 shows the distribution of values of η as obtained from the first 1500 images in the archive. Two obvious features of the distribution are that it has a maximum in the vicinity of $\eta = 0$ and a rather wide distribution giving significant weight to relatively high values of η. The distribution is fitted by the function

$$P(\eta) = \exp\{[C_1 + C_2(\eta - \langle\eta\rangle) + C_3(\eta - \langle\eta\rangle)^2]^{C_4}\} \quad (3)$$

in a log-log scale that emphasizes the distribution ends. A value of $C_4 = 1$ implies a Gaussian distribution. However, its fitted value is approximately equals to 0.5. This implies of a slower decay of the distribution for negative values of η, for which the fit is excellent. The conclusion, therefore, is that the database η is not representative of the image η's. It would be of interest to ask whether there are any specific features in images with high η which are relatively rare compared to images with lower η's. For the benefit of the readers, we have presented a number of images of high η versus a number of images with low η in the web site listed in Ref. [22]. Our conclusion from direct observation of the images is that a large value of eta is usually associated with an image in which the detail seems less pronounced, as expected, of course, from Eq. (2) which defines eta. Having said that, it is also evident that the difference to the eye between images differing considerably in eta is not very striking, and we have found it quite difficult, in some cases, to guess just by looking at a pair of such images that their difference in η is that big.

The wide distribution of the η's implies that a whole family of models is needed to describe scaling in natural images. This is not different from the need to introduce the family of variants of the KPZ equation in response to experimental work [12,23–25] which disagrees with the standard KPZ equation. Most of those variants of KPZ are based on the assumption that the noise, describing the

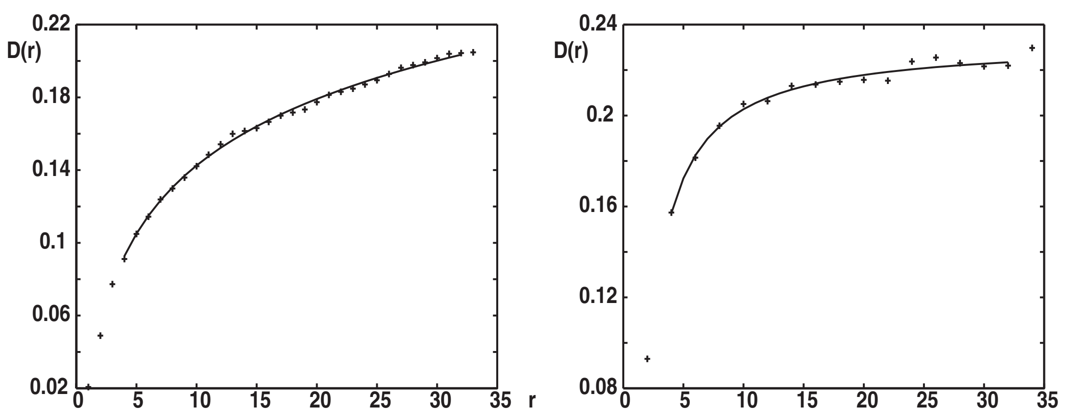

FIG. 3. Left panel: Image 59, $\eta = 0.04$. Right panel: Image 712, $\eta = 1.01$.

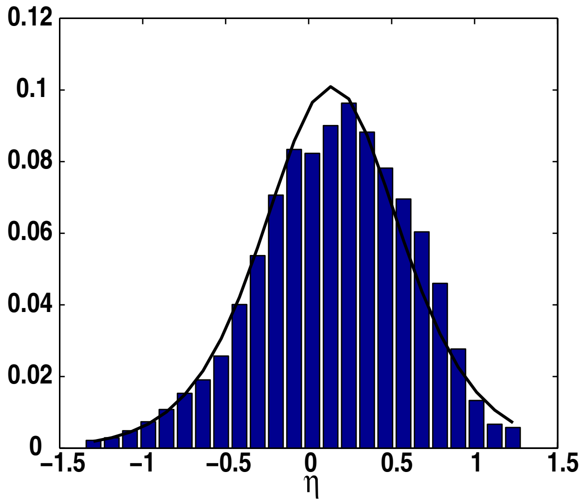

FIG. 4 (color online). Distribution of η for 1500 images in the archive. Fit function: $P(\eta) = \exp\{[C_1 + C_2(\eta - \langle\eta\rangle) + C_3(\eta - \langle\eta\rangle)^2]^{C_4}\}$. $C_4 \approx 0.5$. Mean $= 0.155$; variance $= 0.21$; skewness $= -0.28$; kurtosis $= 2.9$.

random deposition process, is algebraically correlated in either space [26–28] or time [26,29–31]. In any case the scaling exponent depends on the exponent characterizing the decay of the correlation. Is it possible that the spread in the η's in natural images is the result of the action of correlated noises with different algebraic decays? The processes involved in shaping the scene of which an image is taken are very complex and will not be analyzed here. We would like to single out, however, one process that has the required properties, without saying that it is the only such process or even the most important one but that it is enough to make us suspect that various algebraically correlated random processes are responsible for the wide distribution of the η's. Compare the distribution of fallen leaves under calm conditions with their distribution following a turbulent wind. The turbulent wind velocity is random and algebraically correlated. Consequently, we find under calm conditions a random but more or less even coverage of the ground by the leaves, while after a turbulent wind we see clusters on various scales. This may have a definite effect on η. To understand which of the possible different factors might dominate the value of η, we suggest obtaining images of the same location as a function of time and correlating possible changes in the scaling exponents with measurable changing properties of the local environment.

Scaling in natural images remains an open question that needs further experimental and statistical study combined with the construction of physical models that can explain the emergence of such correlations.

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