Four-Loop QCD Corrections to the Electroweak ρ Parameter

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The four-loop QCD corrections to the electroweak ρ parameter arising from top and bottom quark loops are computed. Specifically we evaluate the missing "nonsinglet" piece. Using algebraic methods the amplitude is reduced to a set of around 50 new master integrals which are calculated with various analytical and numerical methods. The inclusion of the newly completed term halves the final value of the four-loop correction for the minimally renormalized top-quark mass. The predictions for the shift of the weak mixing angle and the W-boson mass are thus stabilized.

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Electroweak precision measurements and calculations provide stringent and decisive tests of the quantum fluctuations predicted from quantum field theory. As a most notable example, the indirect determination of the top-quark mass, m_t, mainly through its contribution to the ρ parameter [1], coincides remarkably well with the mass measurement performed by the CDF and D0 experiments at the Fermilab Tevatron [2]. Along the same line, the bounds on the mass of the Higgs boson depend critically on the knowledge of m_t and the control of the top-mass dependent effects on precision observables.

A large group of dominant radiative corrections can be absorbed in the shift of the ρ parameter from its lowest order value $\rho_{\text{Born}} = 1$. The result for the one-loop approximation

$$\delta\rho = 3x_t = 3\frac{G_F m_t^2}{8\sqrt{2}\pi^2}, \tag{1}$$

hence quadratic in m_t, was first evaluated in [3] and used to establish a limit on the mass splitting within one fermion doublet. In order to make full use of the present experimental precision, this one-loop calculation has been improved by two-loop [4–6] and even three-loop QCD corrections [7,8]. Also important are two-loop [9–13] and three-loop [14,15] electroweak effects proportional to x_t^2 and x_t^3, respectively, and the three-loop mixed corrections of order $\alpha_s x_t^2$ [15].

An important ingredient for the interpretation of these results in terms of top-mass measurements performed at hadron colliders or at a future linear collider is the relation between the pole mass and the $\overline{\text{MS}}$-mass definitions, the former being useful for the determination of m_t at colliders, the latter being employed in actual calculations and in short-distance considerations. To match the present three-loop precision of the ρ parameter, this relation must be known in two-loop approximation [16–19], and for the four-loop calculation under discussion the corresponding three-loop result [20–22] must be employed.

For fixed pole mass of the top quark, the three-loop result leads to a shift of about 10 MeV in the mass of the W boson as discussed in [15,23]. (This applies both to the pure QCD corrections and the mixed QCD-electroweak one.) Conversely, the corresponding shift of the top-quark pole mass amounts to 1.5 GeV. (Similar considerations apply to the effective weak mixing angle and other precision observables.) These values are comparable to the experimental precision anticipated for top- and W-mass measurements at the International Linear Collider [24]. In addition, there exists a disagreement (on the level of 3σ) between the values of the so-called on-shell week mixing angle, $\sin^2\theta_W$, as measured by NuTeV Collaboration in deep-ienalstic neutrino scattering [25] and as obtained from the global fit [26] of the standard model to the electroweak precision data [27].

From all these considerations an improvement of the theoretical accuracy seems, therefore, desirable. This, however, requires the evaluation of four-loop QCD corrections to the ρ parameter, the topic of the present work.

The shift in the ρ parameter is given by

$$\delta\rho = \frac{\Pi_T^Z(0)}{M_Z^2} - \frac{\Pi_T^W(0)}{M_W^2}, \tag{2}$$

where $\Pi_T^{W/Z}(0)$ are the transversal parts of the W- and Z-boson self-energies at zero momentum transfer, respectively. The calculation is thus reduced to the evaluation of vacuum (tadpole) diagrams.

The W self-energy receives contributions from the correlator of the "nondiagonal" t-b current only. Contributions to the Z self-energy originate only from the "diagonal" axial current correlator induced by top-quark loops. The vector current part vanishes due to current conservation; the bottom quark is taken as massless. The nonvanishing parts of $\Pi_T^Z(0)$ are conveniently decomposed into nonsinglet and singlet pieces characterized by Feynman diagrams where the external current couples to the same and to two different closed fermion lines, respectively. In three-loop approximation the singlet-piece is larger than the nonsinglet piece by nearly a factor 20. This has motivated the authors of [33] to evaluate, in a first step, the four-loop singlet piece. The strategy employed in that paper was based on the algebraic reduction of the amplitudes to a small set of master-integrals with the help of the Laporta algorithm [34,35], an approach which was recently also applied to the evaluation of the two lowest nonvanishing Taylor coefficients of the vacuum polarization [36,37] and to the decoupling of heavy quarks in QCD [38–40], both in four-loop approximation.

In order to complete the evaluation of the four-loop QCD corrections to the ρ parameter, $\Pi_T^W(0)$ and the nonsinglet parts of $\Pi_T^Z(0)$ are required. The evaluation of $\Pi_T^Z(0)$ is fairly straightforward: the input diagrams have been generated with QGRAF [41]; for the algebraic reduction to master integrals an efficient program has been constructed [42] which relies on FORM3 and FERMAT [43–45]. Furthermore, the full set of the corresponding master integrals is available with high precision [46,47].

The evaluation of $\Pi_T^W(0)$, however, requires the knowledge of a sizeable number of new master integrals, a major part of them (around 40) nontrivial to evaluate precisely. The master integrals can be chosen in many different ways. As discussed in [47] the choice of a so-called "ϵ-finite basis" leads to integrals particularly suited for the evaluation through Padé approximations. On the other hand, topologies with eight lines or less are conveniently calculated through difference equations. In the present Letter we therefore employ a combined approach, which makes use of difference equations [35,48] to evaluate the simpler topologies, i.e., those with up to eight lines, and a seminumerical method based on Padé approximations [47,49–52]. There, a suitably chosen line of the four-loop vacuum diagram is cut, the large- and the small-q^2 behavior of the resulting three-loop propagator are calculated analytically [53–56], the function in the whole region is represented by Padé approximations, and the remaining q^2 integration is performed numerically (see Fig. 1).

An estimate of the numerical uncertainty is obtained by comparing different Padé approximations based on the

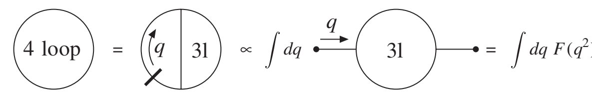

FIG. 1. Symbolic description of the Padé method: one line of the vacuum integral is cut; the resulting propagator is represented by a Padé approximation and integrated numerically.

same input information from the large and small q^2 region, or by increasing the input information through inclusion of more terms from the high and the low q^2 region. Furthermore, in all cases at least two different lines were cut to check the consistency of the results. A detailed discussion of the various applications, characteristic examples, and comparisons with analytic results, e.g., for the lowest moment of the polarization function, can be found in [52].

In contrast to the applications discussed in earlier publications [47,51,52], massless cuts unavoidably arise in some of the relevant diagrams and a generalization of the method is required: in addition to the introduction of a suitably chosen function needed for the subtraction of the high energy logarithms, another function is employed for the subtraction of the logarithms arising from the massless cut in the low energy limit.

The result for the shift in the ρ parameter can be cast into the following form:

$$\delta\rho^{\overline{\rm MS}} = 3x_t \sum_{i=0}^{3}\left(\frac{\alpha_s}{\pi}\right)^i \delta\rho_i^{\overline{\rm MS}}. \quad (3)$$

Here x_t is expressed in terms of the $\overline{\rm MS}$ quark mass $m_t \equiv m_t(\mu)$ at scale $\mu = m_t$, and α_s, defined in the $\overline{\rm MS}$ scheme for six flavors, is chosen at the same scale. The normalization factors are such that $\delta\rho_0^{\overline{\rm MS}} = 1$.

For the four-loop nonsinglet result, decomposed according to the various color structures and the n_f dependence, we find:

$$\delta\rho_3^{\overline{\rm MS}}(\text{nonsinglet}) = 1.5211C_F^3 + 1.2363C_F^2C_A - 2.3132C_F^2Tn_l - 4.5962C_F^2Tn_h + 0.7438C_FC_A^2 - 1.3705C_FC_ATn_l + 2.5037C_FC_ATn_h + 0.4681C_FT^2n_l^2 + 0.6880C_FT^2n_h^2 + 0.8495C_FT^2n_hn_l, \quad (4)$$

with $C_F = (N_c^2 - 1)/(2N_c)$, $C_A = N_c$, and $T = 1/2$, where $N_c = 3$ is the number of colors. n_f denotes the number of active (light plus heavy) quark fields, with $n_f = n_l + n_h$. This result has been also independently obtained with the help of direct application of the Padé approximation method like it was described in [51,52] for the lowest Taylor coefficients of the vacuum polarization. We find agreement for all color structures with the relative accuracy varying between 0.4% and 4%, and confirm the result for the singlet contribution.

Setting $n_h = 1$, $n_l = 5$, and the color coefficients to their natural values, we find

$$\delta\rho_3^{\overline{\rm MS}} = \delta\rho_3^{\overline{\rm MS}}(\text{singlet}) + \delta\rho_3^{\overline{\rm MS}}(\text{nonsinglet}) = -3.2866 + 1.6067 = -1.6799, \quad (5)$$

where we have also displayed the result of [33] for the singlet piece. The singlet piece is still larger than the nonsinglet piece by a factor two. Nevertheless, the hierarchy is less pronounced than in the three-loop case. Numerically, the overall correction looks small, just as in the two- and three-loop case. However, if the result is expressed in terms of the pole mass, a major shift originates from the large correction in the pole-$\overline{\rm MS}$ relation:

$$\delta\rho_3^{\rm pole} = -93.1501. \tag{6}$$

For fixed top mass, this corresponds to a shift of around 2 MeV in the W-boson mass, well below the precision anticipated for future experiments.

In conclusion, the full $\mathcal{O}(X_t\alpha_s^3)$ contribution to the ρ parameter proves to be small and the result based on the three-loop calculation is stabilized.

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Note added.—The results of our calculations have been recently confirmed in the independent work [57].

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