Instability and Evolution of Nonlinearly Interacting Water Waves

P. K. Shukla,[1,2] I. Kourakis,[2] B. Eliasson,[2] M. Marklund,[1] and L. Stenflo[1]

[1]*Centre for Nonlinear Physics, Department of Physics, Umeå University, SE-90187 Umeå, Sweden*

[2]*Institut für Theoretische Physik IV and Centre for Plasma Science and Astrophysics, Fakultät für Physik und Astronomie, Ruhr-Universität Bochum, D-44780 Bochum, Germany*

(Received 16 February 2006; published 30 August 2006)

We consider the modulational instability of nonlinearly interacting two-dimensional waves in deep water, which are described by a pair of two-dimensional coupled nonlinear Schrödinger equations. We derive a nonlinear dispersion relation. The latter is numerically analyzed to obtain the regions and the associated growth rates of the modulational instability. Furthermore, we follow the long term evolution of the latter by means of computer simulations of the governing nonlinear equations and demonstrate the formation of localized coherent wave envelopes. Our results should be useful for understanding the formation and nonlinear propagation characteristics of large-amplitude freak waves in deep water.

DOI: 10.1103/PhysRevLett.97.094501 PACS numbers: 47.35.−i, 92.10.Hm

Recently, there has been much interest [1–5] in investigating the nonlinear formation of "freak" waves (also known as rogue waves, killer waves, or giant waves) in the oceans. Such waves are responsible for the loss of many ships and lives [6]. Ocean waves usually have a spectrum of waves that depends on the variations in the wind speed and direction [7,8]. Freak waves are extraordinarily large-amplitude localized water surface excitations, whose heights exceed many times the wave train height [9]. They may occur both in deep and shallow water, and may be created due to both statistical, linear (geometrical and spatiotemporal focusing), and nonlinear effects [6]. In nonlinear models, these steep objects are created on account of a delicate balance between the nonlinearity of the fluid and wave dispersion. The nonlinear dynamics of a single modulated large-amplitude water wave train can be modeled by a nonlinear Schrödinger equation [10–14], which admits the Benjamin-Feir (modulational) instability [15–18] and the formation of rogue waves [19]. Even though a wide spectrum in the wave field may stabilize a wave train, the Benjamin-Feir instability is quite robust against a narrow spectrum random field [6]; it occurs if the spectral width σ of the wave is less than twice the average steepness (defined as $k_0\bar{a}$ where k_0 is the central wave number and $\bar{a}$ is the rms value of the amplitude) [20]. We stress that collapse of waves is inherently a two-dimensional problem, and that the single wave case has been investigated extensively in Ref. [17]. This indicates that the higher dimensionality of the problem at hand introduces a new flavor to the dynamics of interacting deep water waves, and brings us closer to observational prerequisites for such waves [21].

Onorato *et al.* [3] recently developed a simple weakly nonlinear model for two nonlinearly interacting water waves in deep water with two different directions of propagation. They showed that the dynamics of these coupled waves is governed by two coupled nonlinear Schrödinger (CNLS) equations, and presented an investigation of the modulational instability for a one-dimensional two-wave system. A fourth-order nonlinear evolution equation for two Stoke wave trains in deep water was derived by Dhar and Das [22]. The two-wave system is similar to what has also been investigated in nonlinear optics [23,24], in Bose-Einstein condensates [25,26], in signal transmission lines [27], in nonlinear negative refraction index metamaterials [28,29], and in nonlinear plasmas [30–34].

In this Letter, we shall present for the first time a theory for the modulational instability of a pair of two-dimensional nonlinearly coupled water waves in deep water, as well as the formation and dynamics of localized freak wave packets. For this purpose, we shall use the CNLS equations of Onorato *et al.* [3], which are valid for a system of obliquely propagating waves (crossing sea states). In Ref. [3] the x axis has been defined as the middle (dichotome) between the two directions of propagation, viz. $\mathbf{k}_A = (k_{A,x}, k_{A,y}) \equiv (k, l)$ and $\mathbf{k}_B = (k_{B,x}, k_{B,y}) \equiv (k, -l)$; we shall assume that both k and l are larger than zero. The frequencies ω_j of the two carrier waves (i.e., $j = A, B$) are related to the wave vectors $\mathbf{k}_j$ by the deep water dispersion relation [35] $\omega_j = \sqrt{g|\mathbf{k}_j|}$, where g denotes the gravitational acceleration. We note that the above hypothesis on the two-wave propagation directions implies $\omega_A = \omega_B = \sqrt{g\kappa}$, where we have defined the wave number norm $\kappa \equiv \sqrt{k^2 + l^2}$. Moreover, we investigate the full dynamics of nonlinearly interacting deep water waves subjected to modulational or filamentation instabilities. It is found that random perturbations can grow to form inherently nonlinear water wave structures, the so-called freak waves, through the nonlinear interaction between two coupled water waves. The latter should be of interest for explaining recent observations in water wave dynamics.

Multiplying by i the system of two-dimensional CNLS equations (4) and (5) of Ref. [3], and correcting a misprint [the coefficient in front of the $\partial^2 B/\partial x \partial y$ term in Eq. (5) of Ref. [3]], we have

$$i\left(\frac{\partial A}{\partial t} + C_x\frac{\partial A}{\partial x} + C_y\frac{\partial A}{\partial y}\right) + \alpha\frac{\partial^2 A}{\partial x^2} + \beta\frac{\partial^2 A}{\partial y^2} + \gamma\frac{\partial^2 A}{\partial x\partial y} - \xi|A|^2A - 2\zeta|B|^2A = 0, \quad (1a)$$

and

$$i\left(\frac{\partial B}{\partial t} + C_x\frac{\partial B}{\partial x} - C_y\frac{\partial B}{\partial y}\right) + \alpha\frac{\partial^2 B}{\partial x^2} + \beta\frac{\partial^2 B}{\partial y^2} - \gamma\frac{\partial^2 B}{\partial x\partial y} - \xi|B|^2B - 2\zeta|A|^2B = 0, \quad (1b)$$

where A and B are the amplitudes of the slowly varying wave envelopes such that the surface elevations (in meters) are given by $\eta_A = (1/2)A(\mathbf{r}, t)\exp(ikx + ily - i\omega t) +$ c.c. and $\eta_B = (1/2)B(\mathbf{r}, t)\exp(ikx - ily - i\omega t) +$ c.c., where c.c. denotes complex conjugate. Here, $C_x = \omega k/2\kappa^2$ and $C_y = \omega l/2\kappa^2$ are the group speed components along the x and y axis, respectively, and the group velocity dispersion coefficients are $\alpha = \omega(2l^2 - k^2)/8\kappa^4$, $\beta = \omega(2k^2 - l^2)/8\kappa^4$, and $\gamma = -3\omega lk/4\kappa^4$, and the nonlinearity coefficients are given by $\xi = \omega\kappa^2/2$ and $\zeta = \omega(k^5 - k^3l^2 - 3kl^4 - 2k^4\kappa + 2k^2l^2\kappa + 2l^4\kappa)/2\kappa^2(k - 2\kappa)$ [see Ref. [3]]. It should be pointed out that Eq. (1) was first written down about 40 years ago by Benney and Newell [36] who gave formulas for all the coefficients except ξ and ζ, which, however, can be found in Ref. [3].

It is easy to verify that the system of two-dimensional CNLS equations (1) has the space independent harmonic solutions $A_{eq} = A_0\exp[-i(\xi|A_0|^2 + 2\zeta|B_0|^2)t]$ and $B_{eq} = B_0\exp[-i(\xi|B_0|^2 + 2\zeta|A_0|^2)t]$. Assuming a small (linear) harmonic perturbation around the above mentioned equilibrium states with the wave vector $\mathbf{K} = (K, L)$ and the frequency Ω, i.e., substituting with $A \to (A_0 + \epsilon A_1) \times \exp[-i(\xi|A_0|^2 + 2\zeta|B_0|^2)t]$ and $B \to (B_0 + \epsilon B_1) \times \exp[-i(\xi|B_0|^2 + 2\zeta|A_0|^2)t]$ into (1), linearizing in the small real parameter $\epsilon \ll 1$, then separating the real and imaginary parts, combining the resultant equations, and Fourier transforming, we obtain the nonlinear dispersion relation

$$[(\Omega - C_xK - C_yL)^2 - \Omega_1^2] \times [(\Omega - C_xK + C_yL)^2 - \Omega_2^2] = \Omega_c^4, \quad (2)$$

where $\Omega_1^2 = (\alpha K^2 + \beta L^2 - \gamma KL)(\alpha K^2 + \beta L^2 + \gamma KL + 2\xi|A_0|^2)$, $\Omega_2^2 = (\alpha K^2 + \beta L^2 + \gamma KL)(\alpha K^2 + \beta L^2 - \gamma KL + 2\xi|B_0|^2)$, and $\Omega_c^4 = 16\zeta^2|A_0|^2|B_0|^2(\alpha K^2 + \beta L^2 - \gamma KL)(\alpha K^2 + \beta L^2 + \gamma KL)$. For one-dimensional wave propagation, i.e., for $L = 0$, Eq. (2) is identical to Eq. (11) of Ref. [3]. Onorato *et al.* [3] presented numerical results for the modulational instability regimes and growth rates for $L = 0$ and $A_0 = B_0$ from their Eq. (11). We note that the dispersion relation (2) is nonlinear in the wave amplitudes (but linear in the small expansion parameter).

Taking into account that a measure of the wave nonlinearities are given by the "steepnesses" κA and κB, we will use the natural normalizations $A_0 = A_0'/\kappa$ and $B_0 = B_0'/\kappa$ for the wave amplitudes. For the wave numbers and frequencies we use $K' = K/\kappa$, $L' = L/\kappa$, $k' = k/\kappa$, $l' = l/\kappa$, and $\Omega' = \Omega/\omega$. The coefficients are normalized as $C_x' = C_x\kappa/\omega = k'/2$, $C_y' = C_y\kappa/\omega = l'/2$, $\alpha' = \alpha\kappa^2/\omega = (2l'^2 - k'^2)/8$, $\beta' = \beta\kappa^2/\omega = (2k'^2 - l'^2)/8$, $\gamma' = \gamma\kappa^2/\omega = -3l'k'/4$, $\xi' = \xi/\omega k^2 = 1/2$, and $\zeta' = \zeta/\omega k^2 = [(k')^5 - (k')^3(l')^2 - 3k'(l')^4 - 2(k')^4 + 2(k')^2 \times (l')^2 + 2(l')^4]/2(k' - 2)$, and all quantities in Eq. (2) can be replaced with their primed counterparts. Here we have $k' = \cos\theta$ and $l' = \sin\theta$, where θ is the angle between the wave directions and the dichotome. Except for numerical factors, Eq. (2) then only contains the angle θ, the known wave numbers K' and L', the wave amplitudes A_0' and B_0', and the unknown frequency Ω'.

In the following, we numerically solve our nonlinear dispersion relation (2) and present the growth rate Γ (the imaginary part of Ω) in Figs. 1–3, where we have studied the impact of different angles θ on the growth rates for interacting waves. For all cases we used the wave amplitudes $0.1/\kappa$ for the two waves. In the left-hand panels of Figs. 1 and 2 we show the single wave cases, which exhibit the standard Benjamin-Feir instability [cf. Eq. (3.6) and Fig. 1 of Ref. [18]], tilted by the angle θ in the (K, L) plane. The right-hand panels show the cases of interacting waves. We see from Fig. 1 that a relatively small $\theta = \pi/8$ gives rise to a new instability with a maximum growth rate that is more than twice as large as the ones for the single wave cases, in the direction of the dichotome. For a larger angle $\theta = \pi/4 \approx 0.79$, displayed in Fig. 2, we see that the two waves do not interact to enhance the linear growth rate

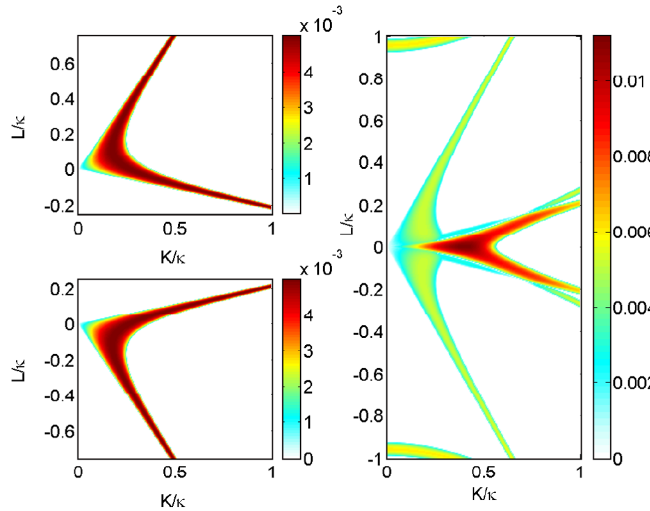

FIG. 1 (color online). The normalized growth rate Γ/ω plotted as a function of K/κ and L/κ. Here we have used $\theta = \pi/8$ and the wave amplitudes $A_0 = B_0 = 0.1/\kappa$. The left upper and lower panels show the cases with a single wave A_0 and B_0, respectively, while the right panel shows the case of interacting waves.

significantly. We note [as pointed out in Ref. [3]] that the coefficient α changes sign when $\theta = \arctan(l/k) = \arctan(1/\sqrt{2}) \approx 0.615$ rad $\approx 35.3°$ so that we have a focusing (defocusing) instability along the x axis for $\theta < 0.615$ (>0.615). This clearly stabilizes the waves so that they only exhibit the standard Benjamin-Feir instability but do not interact to enhance the instability. On the other hand, for two counterpropagating waves with $\theta = \pi/2$, we see in Fig. 3 that there is again an instability along the dichotome [in agreement with Ref. [3]] and also obliquely in narrow bands almost perpendicular to the two waves.

In order to study the dynamics of nonlinearly interacting water wave packets, we solve the coupled system of Eqs. (1). The results are displayed in Fig. 4. In the numerical simulation, we have used the normalization $A' = A/\kappa$, $B' = B/\kappa$, $t' = \omega t$, $x' = \kappa x$, and $y' = \kappa y$ (the other scaled parameters are as above), while the results are shown in dimensional units in Fig. 4. We have used the same parameters as in the right-hand panel of Fig. 1, where the two interacting waves initially have the amplitude $A = B = 0.1/\kappa$, plus low-amplitude noise (random numbers) of the order $10^{-3}/\kappa$ to give a seed for any instability. The wave fronts in Fig. 4 are initially directed primarily in the x direction, reflecting the maximum growth rate in the K direction in the right-hand panel of Fig. 1. The linear growth saturates in the formation of wave packets, localized in the x direction, that are strongly correlated between wave A and B; see the panels at time $t = 900/\omega$. At later times the waves A and B decouple and there are scattered large-amplitude waves that are localized both in the x and y directions. In order to compare with the case of a single wave we have also run a simulation where B was set to zero, so that we have the standard Benjamin-Feir instability shown in the upper left panel of Fig. 1. In this case (not shown here), we could not see the formation of well-defined wave packets, but the instability gave rise to a wide spectrum of waves in different directions, in agreement with the linear analysis in the left-hand panels of Fig. 1. The new instability due to the coupling of the two waves, shown in the right-hand panel of Fig. 1, has a well-defined maximum in the x direction, which is important for the localization of wave energy into localized wave packets seen in Fig. 4. Taking some typical data from ocean waves [7] where a typical wave frequency is 0.09 Hz, we have $\omega = 0.56$ s^{-1}, and $\kappa = \omega^2/g \approx 0.033$ m^{-1}. The waves A and B in Fig. 4 then have the initial amplitudes $|A| = |B| = 0.1/\kappa \approx 3$ m. In Fig. 4, we see at $t = 1200/\omega$ (≈ 670 s) that wave A has some localized wave packets with a maximum amplitude of $\approx 0.35/\kappa \approx 10$ m.

To summarize, we have presented a theoretical study of the modulational instabilities of a pair of nonlinearly interacting two-dimensional waves in deep water, and have shown that the full dynamics of these interacting waves gives rise to localized large-amplitude wave packets. Starting from the CNLS equations of Onorato *et al.* [3], we have derived a nonlinear dispersion equation. The latter has been numerically analyzed to show the dependence of obliqueness of the interacting waves and of the modulations on new classes of modulational instabilities that we found in multidimensional situations. Furthermore, the numerical analysis of the full dynamical system reveals that two water waves can, when nonlinear interactions are taken into account, give rise to novel behavior such as the formation of large-amplitude coherent wave packets with amplitudes more than 3 times the ones of the initial waves.

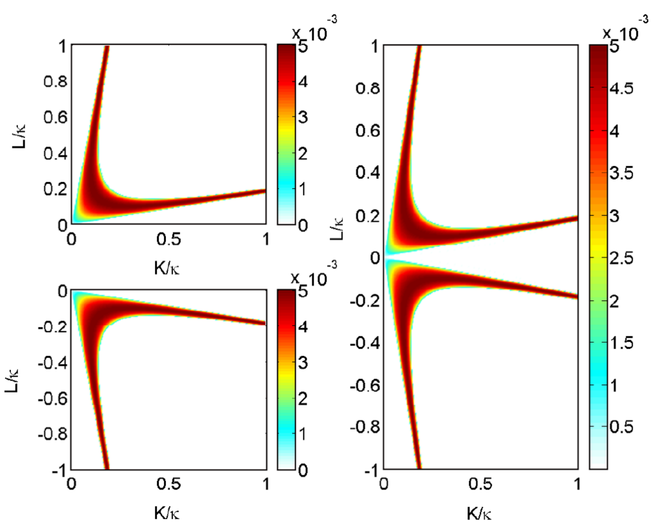

FIG. 2 (color online). The normalized growth rate Γ/ω plotted as a function of K/κ and L/κ. Here we have used $\theta = \pi/4$ and the wave amplitudes $A_0 = B_0 = 0.1/\kappa$. The left upper and lower panels show the cases with a single wave A_0 and B_0, respectively, while the right panel shows the case of interacting waves.

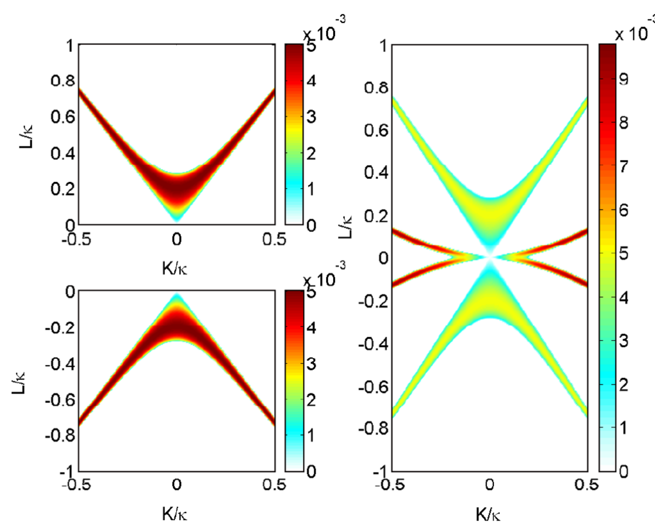

FIG. 3 (color online). The normalized growth rate Γ/ω plotted as a function of K/κ and L/κ. Here we have used $\theta = \pi/2$ and the wave amplitudes $A_0 = B_0 = 0.1/\kappa$. The left upper and lower panels show the cases with a single wave A_0 and B_0, respectively, while the right panel shows the case of interacting waves.

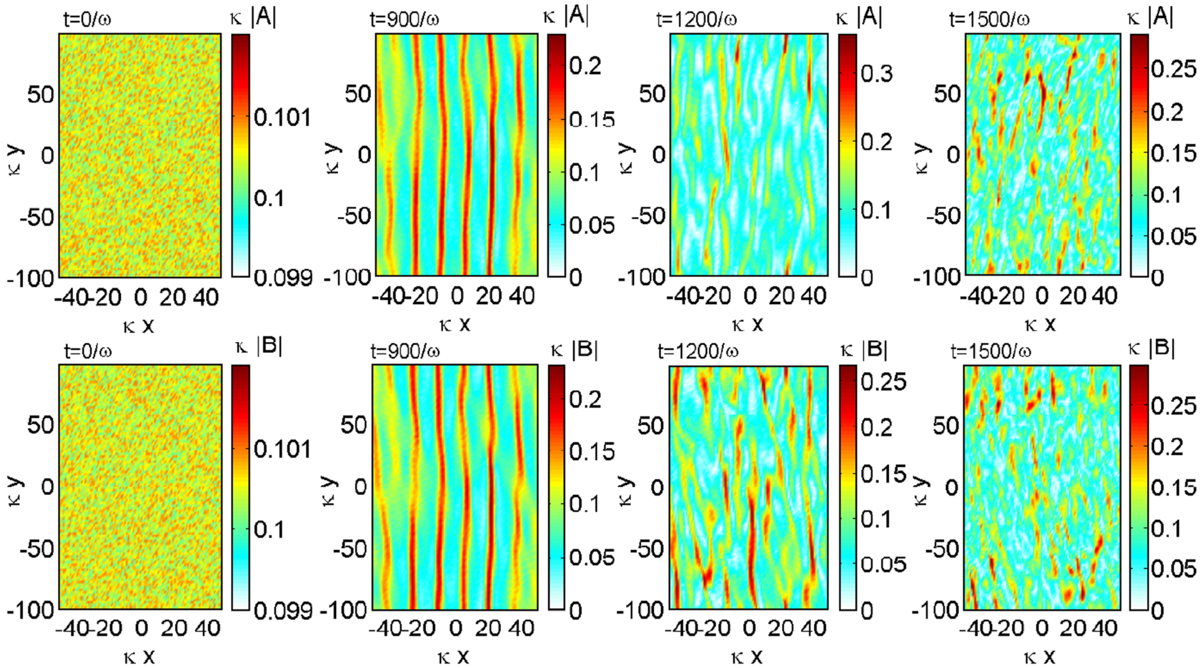

FIG. 4 (color online). The interaction between two waves, initially with equal amplitudes $|A| = |B| = 0.1\kappa^{-1}$ and a propagation angle of $\theta = \pi/8$ relative to the dichotome. Added to the initially homogeneous wave envelopes is a low-amplitude noise of order $10^{-3}/\kappa$ to give a seed to the modulational instability.

This behavior is very different from that of a single wave which experiences the standard Benjamin-Feir instability which dissolves into a wide spectrum of waves. These results will have relevance to the nonlinear instability of colliding water waves, which may interact nonlinearly in a constructive way to produce large-amplitude freak waves in the oceans.

This research was partially supported by the Swedish Research Council and the Deutsche Forschungsgemeinschaft.

[1] M. Onorato *et al.*, Phys. Rev. Lett. **86**, 5831 (2001).
[2] M. Onorato *et al.*, Phys. Rev. Lett. **89**, 144501 (2002).
[3] M. Onorato *et al.*, Phys. Rev. Lett. **96**, 014503 (2006).
[4] A. I. Dyachenko and V. E. Zakharov, JETP Lett. **81**, 255 (2005).
[5] J. L. Hammack *et al.*, J. Fluid Mech. **532**, 1 (2005).
[6] C. Kharif and E. Pelinovsky, Eur. J. Mech. B, Fluids **22**, 603 (2003).
[7] K. Hasselmann *et al.*, J. Phys. Oceanogr. **6**, 200 (1976).
[8] D. E. Hasselmann *et al.*, J. Phys. Oceanogr. **10**, 1264 (1980).
[9] R. G. Dean, *Water Wave Kinematics* (Kluwer, Dordrecht, 1990), pp. 609–612.
[10] V. E. Zakharov, J. Appl. Mech. Tech. Phys. **9**, 190 (1968).
[11] C. Sulem and P.-L. Sulem, *The Nonlinear Schrödinger Equation* (Springer, New York, 1999).
[12] A. R. Osborne *et al.*, Phys. Lett. A **275**, 386 (2000).
[13] R. Fedele, Phys. Scr. **65**, 502 (2002).
[14] R. Fedele and H. Schamel, Eur. Phys. J. B **27**, 313 (2002).
[15] T. B. Benjamin and J. E. Feir, J. Fluid Mech. **27**, 417 (1967).
[16] V. E. Zakharov, Zh. Eksp. Teor. Fiz. **51**, 668 (1966) [Sov. Phys. JETP **24**, 455 (1967)].
[17] V. E. Zakharov *et al.*, Phys. Rep. **398**, 1 (2004).
[18] H. Segur *et al.*, J. Fluid Mech. **539**, 229 (2005).
[19] A. Calini and C. M. Schober, Phys. Lett. A **298**, 335 (2002).
[20] K. B. Dysthe *et al.*, J. Fluid Mech. **478**, 1 (2003).
[21] M. J. Ablowitz *et al.*, Phys. Rev. Lett. **84**, 887 (2000).
[22] A. K. Dhar and K. P. Das, Phys. Fluids A **3**, 3021 (1991).
[23] G. P. Agrawal, Phys. Rev. Lett. **59**, 880 (1987).
[24] L. Berge, Phys. Rev. E **58**, 6606 (1998).
[25] K. Kasamatsu and M. Tsubota, Phys. Rev. Lett. **93**, 100402 (2004).
[26] I. Kourakis *et al.*, Eur. Phys. J. B **46**, 381 (2005).
[27] J. M. Bilbault *et al.*, Phys. Rev. E **51**, 817 (1995).
[28] I. Kourakis and P. K. Shukla, Phys. Rev. E **72**, 016626 (2005).
[29] M. Marklund *et al.*, Phys. Lett. A **341**, 231 (2005).
[30] Y. Inoue, J. Plasma Phys. **16**, 439 (1976).
[31] C. J. McKinstrie and R. Bingham, Phys. Fluids B **1**, 230 (1989).
[32] P. K. Shukla, Phys. Scr. **45**, 618 (1992).
[33] I. Kourakis *et al.*, New J. Phys. **7**, 153 (2005).
[34] S. V. Vladimirov *et al.*, Phys. Lett. A **153**, 144 (1991).
[35] V. I. Karpman, *Nonlinear Waves in Dispersive Media* (Pergamon, New York, 1975), p. 10.
[36] D. J. Benney and A. C. Newell, J. Math. Phys. (N.Y.) **46**, 133 (1967).