Environment-Mediated Control of a Quantum System

Raffaele Romano* and Domenico D'Alessandro†

Department of Mathematics, Iowa State University, Ames, Iowa 50011, USA

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Recently, a new approach for the controllability of a two-dimensional quantum system S has been proposed, based on its interaction with an initially uncorrelated two-dimensional probe P whose initial state can be arbitrarily modified. Following this scheme and considering a particular model for the environment, we show that, in some specific cases, the environment-induced entanglement is rich enough to completely control the dynamics of S. Under suitable conditions on the interaction of S, P, and the environment, we prove that the state of S can be driven to an arbitrary target state by varying the initial state of P.

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Introduction.—In the past decades, a large interest has grown around the quantum theory of information and computation [1]. Because of the peculiar features exhibited by microscopic systems, these are deemed appropriate to be used in information processing not feasible by means of classical systems. The major obstacle towards these applications is, however, the fragility of quantum systems. They are strongly affected by interaction with the external environment, which destroys the desirable features of quantum dynamics, producing irreversibility and decoherence. A number of ideas from control theory have been introduced to overcome this problem, as, for example, in the quantum theory of *feedback* [2,3]. Control theoretical ideas are also used in the analysis of quantum dynamics and generation of entanglement [4–8] and in the development of algorithms for the control of quantum systems [9–12].

In the context of control theory of quantum systems, a fundamental question is to what extent it is possible to modify the state of a system by an external action. Although the (in principle, uncontrollable) interaction with the environment usually has a negative impact on the possibility of manipulating the quantum state, a recent investigation in Ref. [13] has shown that the interaction with a common environment can generate entanglement for a couple of systems plunged in it. Motivated by this result, we present in this Letter a new approach to the control of a quantum system interacting with its surroundings, in which the environment-induced entanglement is used to control the system *despite decoherence*. Therefore, in this model, the environment represents a positive factor.

In the following, it is assumed that the dynamics of a quantum system S depends on a number of parameters u (the *controls*) that can be externally modified, i.e.,

$$\rho_S(t, u) = \gamma(t, u)[\rho_S(0)] \tag{1}$$

for some linear map $\gamma = \gamma(t, u)$, where ρ_S is the state of S. In the controllability analysis, one wishes to study the state transitions that can be induced by choosing the controls u. To formalize this question, one introduces the *reachable set from ρ_S at time t* as the set of all the states that can be reached from ρ_S by appropriately varying the control, i.e.,

$$\mathcal{R}(\rho_S, t) = \{\rho_S(t, u) | \rho_S(0) = \rho_S, u \in \mathcal{U}\}, \tag{2}$$

where $\mathcal{U}$ is the set of admissible controls. The *reachable set from ρ_S* is given by

$$\mathcal{R}(\rho_S) = \lim_{T\to+\infty} \mathcal{R}_T(\rho_S), \tag{3}$$

where

$$\mathcal{R}_T(\rho_S) = \bigcup_{0\le t\le T} \mathcal{R}(\rho_S, t) \tag{4}$$

is the reachable set from ρ_S until T. In general, $\mathcal{R}(\rho_S) \subseteq \mathcal{P}_S$, where $\mathcal{P}_S$ is the convex set of all density matrices associated with S. The main controllability properties are defined in terms of these sets. In particular, the system S is said to be *accessible* if and only if $\mathcal{R}_T(\rho_S)$ contains a nonempty open set of $\mathcal{P}_S$ for all T and for all $\rho_S \in \mathcal{P}_S$. From a physical point of view, this means that it is possible to move every initial state ρ_S in any direction in $\mathcal{P}_S$ by choosing suitable control parameters u. Moreover, the system S is said to be *controllable* if and only if $\mathcal{R}(\rho_S) = \mathcal{P}_S$ for all initial states $\rho_S \in \mathcal{P}_S$. Consequently, for a controllable system S, every transition between any two states in $\mathcal{P}_S$ is allowed.

Controllability has been investigated in depth for quantum systems when the controls u appear as parameters in the Hamiltonian of the system. This study has concerned both unitary and dissipative evolutions and has led to several algebraic criteria to test controllability (see, e.g., [14–17]). Techniques of control which use tunable parameters in the Hamiltonian of the systems are referred to as coherent control methods since the control directly affects the coherent part of the dynamics, for both unitary or dissipative evolutions. A different scenario, motivated by several experimental setups, is where the control variables affect an auxiliary system which then interacts with the quantum system to obtain control [18–20]. In this approach, the controls do not directly affect the evolution of the system through the coherent part of its dynamics.

This is the setting considered here. The system S is allowed to interact with a second system P, called the *probe*, and initially they are in an uncorrelated state $\rho_S \otimes \rho_P$. It is assumed that it is possible to modify the initial state of P before the interaction; therefore, in this case the controls enter the dynamics of S through $\rho_P = \rho_P(u)$.

Controllability and accessibility of S in this new control setting have been investigated under the assumption that the composite system $T = S + P$ is closed. In this case, the dynamics (1) is given by

$$\rho_S(t, u) = \mathrm{Tr}_P(X(t)\rho_S \otimes \rho_P(u)X^\dagger(t)), \tag{5}$$

where Tr_P is the partial trace over the degrees of freedom of P, $X(t) = e^{-iH_T t}$ is the unitary propagator, and $H_T = H_S + H_P + H_{SP}$ is the Hamiltonian of T. The coupling between S and P is given by the interaction Hamiltonian H_{SP}. Necessary and sufficient conditions for controllability and accessibility have been derived in the case of two-dimensional S and P [20], under the hypothesis that it is possible to obtain all the pure states of ρ_P by an arbitrary choice of the control. In particular, the system S evolving under (5) is controllable if and only if there is a time t at which the unitary evolution of the composite system $X(t)$ is locally equivalent to the SWAP operator. Equivalent algebraic conditions of controllability and accessibility can be given by considering the Cartan decomposition [21] of $X(t)$.

The assumption that T is a closed system is valid only in first approximation. In general, there will be an external environment E interacting with T and, thus, affecting the controllability properties of S. In the following, we shall consider a standard model for the environment given by a large number of decoupled harmonic oscillators and show that, for appropriate forms of the bath-system interaction, we can have accessibility and controllability of a system which would be neither controllable nor accessible as a closed system. In fact, system S and probe P are assumed not to be directly interacting and their interaction is completely due to the presence of the environment. In this respect, the case treated is the opposite of the one treated in Ref. [20], where the system and the probe were assumed interacting and no environment was present.

A model of two systems interacting through the environment.—We consider the model of the environment E described in Ref. [22]. E is given by a set of N decoupled harmonic oscillators with Hamiltonian

$$H_E = \sum_{i=1}^{N} \hbar\omega_i\left(b_i^\dagger b_i + \frac{1}{2}\right), \tag{6}$$

where $b_i^\dagger$, b_i are the creation and annihilation operators, respectively, associated to the ith oscillator and ω_i its angular frequency. This is the bosonic bath model as $N \to \infty$, and the considerations on controllability that will follow do not depend on N. We assume $H_T = 0$; that is, the composite system of system and probe $T = S + P$ has no free evolution. We assume a linear coupling between E and T depending on the positions of the oscillators,

$$H_{ET} = \sum_{i=1}^{N} A_T \otimes g_i(b_i + b_i^\dagger), \tag{7}$$

where g_i is the coupling constant of the ith oscillator and A_T an arbitrary Hermitian operator in the Hilbert space of T. The evolution of a state of S is given by

$$\rho_S(t, u) = \mathrm{Tr}_P\,\mathrm{Tr}_E(X(t)\rho_S \otimes \rho_P(u) \otimes \rho_E X^\dagger(t)), \tag{8}$$

where $X(t) = e^{-i(H_E + H_{ET})t}$ and S, P, and E are all initially decoupled. The environment is in the thermal state ρ_E. A_T is a constant of motion since $[A_T, H_E + H_{ET}] = 0$; therefore, it is possible to find the exact analytical expression of the dynamics. It is convenient to introduce the eigenvalues and eigenvectors of A_T, $A_T|\alpha_i\rangle = \alpha_i|\alpha_i\rangle$ for $i = 1, \ldots, 4$. Therefore, (8) becomes

$$\rho_S(t, u) = \sum_{i,j=1}^{4} \mathrm{Tr}_P|\alpha_i\rangle\langle\alpha_j|(\rho_S \otimes \rho_P(u))_{ij}\gamma_{ij}(t), \tag{9}$$

where we introduced the functions

$$\gamma_{ij}(t) = e^{-(\alpha_i - \alpha_j)^2 f(t) + i(\alpha_i^2 - \alpha_j^2)\varphi(t)} \tag{10}$$

and

$$\begin{aligned} f(t) &= \sum_{i=1}^{N}\left(\frac{g_i}{\hbar\omega_i}\right)^2(1 + 2\bar{n}_i)(1 - \cos\omega_i t), \\ \varphi(t) &= \sum_{i=1}^{N}\left(\frac{g_i}{\hbar\omega_i}\right)^2(\omega_i t - \sin\omega_i t), \end{aligned} \tag{11}$$

with $\bar{n}_i$ the average thermal occupation number for the ith oscillator [22].

To compute the partial trace in (9), we need to make some assumptions on the eigenvectors of A_T. We find it convenient to explore two opposite cases: the case when all the eigenvectors are factorized states in the Hilbert space of $S + P$ and the case when they are maximally entangled states. By exploring these two extreme cases, we will find examples of evolutions that are not accessible, accessible but not controllable, or controllable. This last case will prove our claim that the environment induces controllability.

Controllability and accessibility properties.—We first consider the case where the eigenvectors of A_T, $|\alpha_i\rangle$, are factorized states, i.e., $|\alpha_i\rangle = |\alpha_k^S\rangle \otimes |\alpha_l^P\rangle$, with $i = (k, l)$, $k, l = 1, 2$, and the sets $\{|\alpha_1^S\rangle, |\alpha_2^S\rangle\}$ and $\{|\alpha_1^P\rangle, |\alpha_2^P\rangle\}$ are orthonormal bases in the Hilbert spaces of S and P, respectively. In this case,

$$\mathrm{Tr}_P|\alpha_i\rangle\langle\alpha_j| = \delta_{ln}|\alpha_k^S\rangle\langle\alpha_m^S|, \tag{12}$$

and, moreover,

$$(\rho_S \otimes \rho_P(u))_{ij} = (\rho_S)_{km}(\rho_P(u))_{ln}, \tag{13}$$

where $i = (k, l)$ and $j = (m, n)$. Thus, Eq. (9) becomes

$$\rho_S(t, u) = \sum_{k,m=1}^{2}\Big[(\rho_S)_{km}|\alpha_k^S\rangle\langle\alpha_m^S| \times \sum_{n=1}^{2}(\rho_P(u))_{nn}\gamma_{(k,n)(m,n)}(t)\Big], \tag{14}$$

and initial states ρ_S, diagonal in this basis, do not evolve. Examples of evolutions displaying this behavior are determined by interaction terms of the form $A_T = A_S + A_P$ or $A_T = A_S \otimes A_P$, where A_S and A_P are Hermitian operators acting on the Hilbert spaces of S and P. It follows that, in these cases, S is neither accessible nor controllable; therefore, a necessary condition for accessibility and controllability is that at least one eigenvector of A_T is an entangled state in $S + P$.

We assume now that all the eigenvectors are maximally entangled states, i.e., Bell states

$$\begin{aligned} |\alpha_{1,2}\rangle &= \frac{1}{\sqrt{2}}(|\alpha_1^S\rangle \otimes |\alpha_1^P\rangle \pm |\alpha_2^S\rangle \otimes |\alpha_2^P\rangle), \\ |\alpha_{3,4}\rangle &= \frac{1}{\sqrt{2}}(|\alpha_1^S\rangle \otimes |\alpha_2^P\rangle \pm |\alpha_2^S\rangle \otimes |\alpha_1^P\rangle) \end{aligned} \tag{15}$$

in suitable bases $\{|\alpha_1^S\rangle, |\alpha_2^S\rangle\}$ and $\{|\alpha_1^P\rangle, |\alpha_2^P\rangle\}$. It is convenient to use a coherence vector representation for the states in S and P; that is,

$$\rho_S = \tfrac{1}{2}(\mathbb{I} + \vec{s}\cdot\vec{\sigma}^S), \qquad \rho_P = \tfrac{1}{2}(\mathbb{I} + \vec{p}\cdot\vec{\sigma}^P), \tag{16}$$

where $\vec{s}$ and $\vec{p}$ are real vectors in the Bloch spheres of S and P and $\vec{\sigma}^{S,P}$ are the vectors of the Pauli matrices in S and P. In this representation, the dynamics (9) takes the form

$$\vec{s}(t, u) = A(t, \vec{s}_0)\vec{p}(u) + \vec{a}(t, \vec{s}_0), \tag{17}$$

where $A(t, \vec{s}_0)$ is the matrix

$$\frac{1}{2}\,\mathrm{Im}\begin{pmatrix} i\gamma_{13-24}(t) & s_z\gamma_{13-24}(t) & s_y\gamma_{13+24}(t) \\ s_z\gamma_{14-23}(t) & i\gamma_{23-14}(t) & -s_x\gamma_{23+14}(t) \\ -s_y\gamma_{12+34}(t) & s_x\gamma_{34-12}(t) & i\gamma_{12-34}(t) \end{pmatrix} \tag{18}$$

and

$$\vec{a}(t, \vec{s}_0) = \frac{1}{2}\,\mathrm{Re}\begin{pmatrix} s_x\gamma_{13+24}(t) \\ s_y\gamma_{23+14}(t) \\ s_z\gamma_{12+34}(t) \end{pmatrix}. \tag{19}$$

Here we have introduced the notation

$$\gamma_{ij\pm kl}(t) = \gamma_{ij}(t) \pm \gamma_{kl}(t), \tag{20}$$

where the γ_{ij} have been defined in (10), and $\vec{s}_0 = (s_x, s_y, s_z)$ represents the initial state ρ_S.

Assuming that the initial state $\rho_P(u)$ can be an arbitrary state in the Bloch sphere of P, it follows that, in the coherence vector formalism, $\mathcal{R}(\rho_S, t)$ is an ellipsoid contained in the Bloch sphere of S, centered in $\vec{a}(t, \vec{s}_0)$, with the semiaxes given by the singular values of $A(t, \vec{s}_0)$. A sufficient condition for accessibility which is generically satisfied can be given in terms of the eigenvalues of A_T, α_i, $i = 1, \ldots, 4$. In particular, the system is accessible if

$$\begin{aligned} (\alpha_2 - \alpha_4)^2 &\neq (\alpha_1 - \alpha_3)^2, \\ (\alpha_1 - \alpha_4)^2 &\neq (\alpha_2 - \alpha_3)^2, \\ (\alpha_3 - \alpha_4)^2 &\neq (\alpha_1 - \alpha_2)^2. \end{aligned} \tag{21}$$

To see that this is a sufficient condition for accessibility, one calculates the 6th derivative with respect to time of the determinant of the matrix $A(t, \vec{s}_0)$ in (18) for $t = 0$ (all the lower order derivatives are zero). If condition (21) is verified, then this derivative is different from zero. If the system were not accessible, then the matrix $A(t, \vec{s}_0)$ would have a singular value equal to zero for every t in an arbitrarily small interval $[0, \epsilon)$. Therefore, $\det A(t, \vec{s}_0) \equiv 0$ for $t \in [0, \epsilon)$ and its derivatives would all vanish at $t = 0$.

To find conditions for controllability, we look at the evolution of the reachable set. This can be quite involved, since the condition $\mathcal{R}(\rho_S) = \mathcal{P}_S$ depends on the parameters of the model in a nontrivial way. However, our goal is to show that it is possible to use the environment to have complete control over the state of the system. We will exhibit an explicit example. The simplest cases arise when $\mathcal{R}(\rho_S, \hat{t}) = \mathcal{P}_S$ at some time $\hat{t}$. This can be obtained by choosing $\alpha_1 = \alpha_2 = \alpha_3 = 0$ and $\alpha_4 \neq 0$. Since relations (21) are satisfied, we have accessibility for all $\alpha_4 \neq 0$. Moreover, Eqs. (18) and (19) simplify to

$$A(t, \vec{s}_0) = \frac{1}{2}\begin{pmatrix} 1 - \gamma_r(t) & -s_z\gamma_i(t) & s_y\gamma_i(t) \\ s_z\gamma_i(t) & 1 - \gamma_r(t) & -s_x\gamma_i(t) \\ -s_y\gamma_i(t) & s_x\gamma_i(t) & 1 - \gamma_r(t) \end{pmatrix} \tag{22}$$

and

$$\vec{a}(t, \vec{s}_0) = \frac{1}{2}(1 + \gamma_r(t))\begin{pmatrix} s_x \\ s_y \\ s_z \end{pmatrix}, \tag{23}$$

where

$$\begin{aligned} \gamma_r(t) &= e^{-\alpha_4^2 f(t)}\cos(\alpha_4^2\varphi(t)), \\ \gamma_i(t) &= e^{-\alpha_4^2 f(t)}\sin(\alpha_4^2\varphi(t)). \end{aligned} \tag{24}$$

A sufficient condition for controllability is that $A(\hat{t}, \vec{s}_0) = \mathbb{I}$ and $a(\hat{t}, \vec{s}_0) = \vec{0}$ at some time $\hat{t}$ which is obtained if $\gamma_r(\hat{t}) = -1$ and $\gamma_i(\hat{t}) = 0$; that is,

$$\alpha_4^2 f(\hat{t}) = 0, \qquad \alpha_4^2\varphi(\hat{t}) = (2k_1 + 1)\pi, \qquad k_1 \in \mathbb{Z}. \tag{25}$$

The first condition in (25) is satisfied if and only if $\cos\omega_i\hat{t} = 1$; that is, $\omega_i\hat{t} = 2k_{2i}\pi$ for all $i = 1, \ldots N$, with $k_{2i} \in \mathbb{Z}$. Using $\sin\omega_i\hat{t} = 0$, for all $i = 1, \ldots N$, in the second equation, we find a condition on α_4:

$$\frac{1}{\alpha_4^2} = \frac{2}{2k_1 + 1} \sum_{i=1}^{N} k_{2i} \left(\frac{g_i}{\hbar \omega_i} \right)^2, \qquad (26)$$

with arbitrary $k_1 \in \mathbb{Z}$. Therefore, controllability can be achieved for an appropriate combination of the parameters defining the dynamics of the bath (the frequencies ω_i) and the parameters defining the interaction (the α_j's, $j = 1, \ldots, 4$).

Several physical systems are good candidates for the implementation of the environment-mediated control protocol. Following Ref. [13], we propose a pair of quantum dots in a conducting cavity, but spin- or atom-based qubit systems could be considered instead. One dot plays the role of P and it is provided by a double well whose potential can be externally tuned, so ρ_P can be arbitrarily prepared; the other dot is the target system S. An electrostatic barrier between the quantum dots is used to decouple them, $H_{SP} \approx 0$. A small but not vanishing value of H_{SP} justifies a perturbative expansion of the interaction Hamiltonian, leading to an effective interaction with three-body contributions, a necessary condition for having entangled eigenvectors for A_T. In this description, two-body interactions between S and P, in general, appear; however, these terms do not represent an obstacle to the realization of the control protocol even if they have been previously excluded. In fact, they have a positive impact on the controllability of the system, since they increase the entangling capability between S and P and then the ability of steering S through P. However, they complicate the formal treatment. The environment E is given by the electromagnetic field in the cavity, whose intensity can be modified to get optimal conditions [23]. Notice that, for general settings, it is not always possible to modify the environment parameters in order to satisfy the conditions (21) and (26).

Conclusions.—We have described a scheme of indirect control of a quantum system S by means of a probe P, in the presence of a common environment E. We have assumed that S and P do not evolve in the absence of the environment, so that their dynamics is only due to the interaction with E. In this framework, we have proved that the induced correlations between S and P are, in some cases, rich enough to allow *total* control of S through P. These results complement recent research on the creation of entanglement. They suggest that further investigation of the control of a quantum system through its correlations with the environment will prove fruitful.

The model studied here is of interest as a methodology for contrasting the irreversible and decohering action of the environment, for several reasons. It is simple, since it consists of an open-loop procedure: preparation of P, interaction P-S, and extraction of the state of S. It enables state purifications that are not obtainable in the open-loop coherent control framework [15]. Moreover, numerical computations show that the reachable sets are larger than the ones in the coherent control case, even if the system is not controllable. In this approach, the effect of the environment is not merely suppressed but is constructively used to achieve the goal of control.

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*Electronic address: rromano@iastate.edu
†Electronic address: daless@iastate.edu

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[23] It can be proved that condition (26) defines a family of parameters (α_4, ω_i, g_i) that is dense in the space of these parameters; therefore, our protocol does not require a preferred range for the cavity parameters, and standard values of them can be considered.