# "Universal" Distribution of Interearthquake Times Explained

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We propose a simple theory for the "universal" scaling law previously reported for the distributions of waiting times between earthquakes. It is based on a largely used benchmark model of seismicity, which just assumes no difference in the physics of foreshocks, mainshocks, and aftershocks. Our theoretical calculations provide good fits to the data and show that universality is only approximate. We conclude that the distributions of interevent times do not reveal more information than what is already known from the Gutenberg-Richter and the Omori power laws. Our results reinforce the view that triggering earthquakes by other earthquakes is a key physical mechanism to understand seismicity.

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Understanding the space-time-magnitude organization of earthquakes remains one of the major unsolved problems in the physics of the Earth. Earthquakes are characterized by a wealth of power laws, among them, (i) the Gutenberg-Richter distribution $\sim 1/E^{1+\beta}$ (with $\beta \approx 2/3$) of earthquake energies $E$ [1]; (ii) the Omori law $\sim 1/t^p$ (with $p \approx 1$ for large earthquakes) of the rate of aftershocks as a function of time $t$ since a mainshock [2]; (iii) the productivity law $\sim E^a$ (with $a \lesssim 2/3$) giving the number of earthquakes triggered by an event of energy $E$ [3]; (iv) the power law distribution $\sim 1/L^2$ of fault lengths $L$ [4]; and (v) the fractal (and even probably multifractal [5]) structure of fault networks [6] and of the set of earthquake epicenters [7]. From an analysis of the probability density functions (PDF) of waiting times between earthquakes in a hierarchy of spatial domain sizes and magnitudes in southern California, Bak *et al.* discussed in 2002 a unified scaling law combining the Gutenberg-Richter law, the Omori law, and the fractal distribution law in a single framework [8]. This global approach was later refined and extended by the analysis of many different regions of the world by Corral, who proposed the existence of a universal scaling law for the PDF $H(\tau)$ of recurrence times (or interevent times) $\tau$ between earthquakes in a given region $\mathcal{S}$ [9,10]:

$$H(\tau) \simeq \lambda f(\lambda \tau). \tag{1}$$

The remarkable finding is that the function $f(x)$, which exhibits different power law regimes with crossovers, is found almost the same for many different seismic regions, suggesting universality. The specificity of a given region seems to be completely captured solely by the average rate $\lambda$ of observable events in that region, which fixes the only relevant characteristic time $1/\lambda$.

The common interpretation is that the scaling law (1) reveals a complex spatiotemporal organization of seismicity, which can be viewed as an intermittent flow of energy released within a self-organized (critical?) system [11], for which concepts and tools from the theory of critical phenomena can be applied [12]. Beyond these general considerations, there is no theoretical understanding for (1). Under very weak and general conditions, Molchan proved mathematically that the only possible form for $f(x)$, if universality holds, is the exponential function [13], in strong disagreement with observations. Recently, from a reanalysis of the seismicity of southern California, Molchan and Kronrod [14] have shown that the unified scaling law (1) is incompatible with multifractality, which seems to offer a better description of the data.

Here, our goal is to provide a simple theory, which clarifies the status of (1), based on a largely studied benchmark model of seismicity, called the epidemic-type aftershock sequence (ETAS) model of triggered seismicity [15] and whose main statistical properties are reviewed in [16]. The ETAS model treats all earthquakes on the same footing and there is no distinction between foreshocks, mainshocks, and aftershocks: each earthquake is assumed capable of triggering other earthquakes according to the three basic laws (i)–(iii) mentioned above. The ETAS model assumes that earthquake magnitudes are statistically independent and drawn from the Gutenberg-Richter distribution $Q(m)$. Expressed in earthquake magnitudes $m \propto (2/3)\ln_{10} E$, the probability $Q(m)$ for events magnitudes $m_i$ to exceed a given value $m$ is $Q(m) = 10^{-b(m-m_0)}$, where $b \simeq 1$ and $m_0$ is the smallest magnitude of triggering events. We also parametrize the (bare) Omori law for the rate of triggered events of first generation from a given earthquake as $\Phi(t) = \theta c^{\theta}/(c+t)^{1+\theta}$, with $\theta \gtrsim 0$. $\Phi(t)$ can be interpreted as the PDF of random times of independently occurring first-generation aftershocks triggered by some mainshock which occurred at the origin of time $t = 0$. Several authors have shown that the ETAS model provides a good description of many of the regularities of seismicity (see for instance Ref. [17] and references therein).

Our main result is the theoretical prediction (11) below, which is used to fit Corral's data in Fig. 1, with good

 

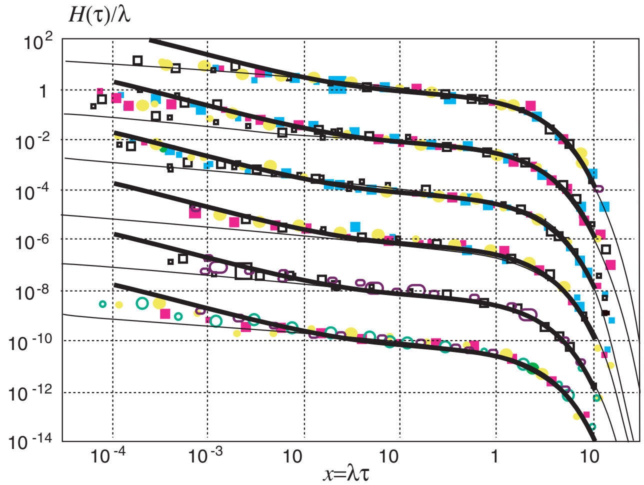


FIG. 1 (color online). Taken from Corral's Ref. [10], plotting the scaled [according to (1)] PDF of the recurrence times $\tau$ between successive earthquakes in various regions of the world, scaled by their corresponding seismicity rates $\lambda$. The PDFs have been translated for clarity. The thin continuous lines are Corral's fits (12) while the thick continuous lines are our prediction (11) based on ETAS model with the parameters $\theta = 0.03$, $n = 0.9$, $a = 0.76$, and $\rho = 1$.

agreement. According to Occam's razor, this suggests that the previously mentioned results on universal scaling laws of interevent times do not reveal more information than what is already captured by the well-known laws (i)–(iii) of seismicity (Gutenberg-Richter, Omori, essentially), together with the assumption that all earthquakes are similar (no distinction between foreshocks, mainshocks, and aftershocks [18]), which is the key ingredient of the ETAS model. Our theory is able to account quantitatively for the empirical power laws found by Corral, showing that they result from subtle crossovers rather than being genuine asymptotic scaling laws. We also show that universality does not strictly hold.

Our strategy to obtain these results is to first calculate the PDF of the number of events in finite space-time windows [17], using the technology of generating probability functions (GPF), which is particularly suitable to deal with the ETAS as it is a conditional branching process. We then determine the probability for the absence of earthquakes in a given time window from which, using the theory of point processes, is determined the PDF of inter-event times. Our analysis is based on the previous calculations of Ref. [17], which showed that, for large areas ($L\sim$ tens of kilometers or more), one may neglect the impact of aftershocks triggered by events that occurred outside the considered space window, while only considering the events within the space domain which are triggered by sources also within the domain.

*Generating probability functions of the statistics of event numbers.*—Consider the statistics of the number $R(t, \tau)$ of events within a time window $[t, t+\tau]$. It is efficiently described by the method of GPF, defined by $\Theta_s(z, \tau) = \langle z^{R(t,\tau)} \rangle$, where the angular brackets denote a statistical average over all possible realizations weighted by their corresponding probabilities. We consider a statistically stationary process, so that $\Theta_s(z, \tau)$ does not depend on the current time $t$ but only on the window duration $\tau$. For the ETAS model, statistical stationarity is ensured by the two conditions that (i) the branching ratio $n$ (or average number of earthquakes or aftershocks of first generation per earthquake) be less than 1 and (ii) the average rate $\omega$ of the Poissonian distribution of spontaneous events be nonzero. The GPF $\Theta_s(z, \tau)$ can then be obtained as [17]

$$\Theta_s(z, \tau) = \exp\Big(-\omega \int_0^\infty [1 - \Theta(z, t, \tau)]dt - \omega \int_0^\tau [1 - z\Theta(z, t)]dt\Big), \qquad (2)$$

where $\Theta(z, t, \tau)$ is the GPF of the number of aftershocks triggered inside the window $[t, t+\tau]$ ($t > 0$) by a single isolated mainshock which occurred at time 0 and $\Theta(z, \tau) = \Theta(z, t = 0, \tau)$. The first (respectively, second) term in the exponential in (2) describes the contribution of aftershocks triggered by spontaneous events occurring before (respectively, within) the window $[t, t+\tau]$.

Ref. [17] previously showed that $\Theta(z, t, \tau)$ is given by

$$\Theta(z, t, \tau) = G[1 - \Psi(z, t, \tau)], \qquad (3)$$

where $G(z)$ is the GPF of the number of first-generation aftershocks triggered by some mainshock, and the auxiliary function $\Psi(z, t, \tau)$ satisfies to

$$\Psi(z, t, \tau) = [1 - \Theta(z, t, \tau)] \otimes \Phi(t) + [1 - z\Theta(z, \tau)] \otimes \Phi(t+\tau). \qquad (4)$$

The symbol $\otimes$ denotes the convolution operator. Integrating (4) with respect to $t$ yields $\int_0^\infty \Psi(z, t, \tau)dt = \int_0^\infty [1 - \Theta(z, t, \tau)]dt + [1 - z\Theta(z, \tau)] \otimes a(\tau)$, so that expression (2) becomes

$$\Theta_s(z, \tau) = \exp\Big[-\omega \int_0^\infty \Psi(z, t, \tau)dt - \omega[1 - z\Theta(z, \tau)] \otimes b(\tau)\Big], \qquad (5)$$

where $b(t) = \int_0^t \Phi(t')dt'$ and $a(t) = 1 - b(t) = \frac{c^\theta}{(c+t)^\theta}$.

*Probability of absence of events.*—For our purpose, the probability $P_s(\tau)$ that there are no earthquakes in a given time window of duration $\tau$ provides an intuitive and powerful approach. It is given by

$$P_0(\tau) \equiv \Theta_s(z = 0, \tau) = \exp\Big[-\omega \int_0^\infty \Psi(t, \tau)dt - \omega\tau + \omega A(\tau)\Big], \qquad (6)$$

where $\Psi(t, \tau) = \Psi(z = 0, t, \tau)$ and $A(\tau) = \int_0^\tau a(t)dt \simeq \frac{c}{1-\theta}(\tau/c)^{1-\theta}$, for $\tau \gg c$.

To make progress in solving (3)–(5), let us expand $G(z)$ in powers of $z$:

$$G(z) = 1 - n + nz + B(1 - z)^\gamma + \ldots, \qquad (7)$$

where $\gamma = b/\alpha$ [where $\alpha = (3/2)a < 1$ is the productivity exponent when using magnitudes] and $B = n\Gamma(2-\gamma)(\gamma-1)^{\gamma-1}/\gamma^{\gamma}$. While we can calculate the looked-for distribution of recurrence times using the shown expansion up to order $(1-z)^{\gamma}$, it turns out that truncating (7) at the linear order is sufficient to explain quantitatively Corral's results, as we show below. Using $G(z) = 1 - n + nz$ has the physical meaning that each earthquake is supposed to generate at most one first-generation event (which does not prevent it from having many aftershocks when summing over all generations). Indeed, interpreted in probabilistic terms, $G(z) = 1 - n + nz$ says that any earthquake has the probability $1 - n$ to give no offspring and the probability $n$ to give one aftershock (of first generation). This linear approximation is bound to fail for small recurrence times associated with the large productivity of big earthquakes and, indeed, we observe some deviations for the shorter recurrence times below several minutes as discussed below. The linear approximation is not intended to describe the statistics of very small recurrence times within clusters of events triggered by large mainshocks, but is approprioate for "quiet" periods of seismic activity.

The linear approximation bypasses much of the complexity of the nonlinear integral equations (3) and (4) to obtain $\int_0^{\infty}\Psi(t,\tau)dt = \frac{A(\tau)}{1-n}$. Expression (6) becomes (for $\tau \gg c$)

$$P_s(\tau) = \exp\left[-(1-n)x - \frac{na\rho^{\theta}}{1-\theta}x^{1-\theta}\right], \quad (8)$$

where

$$x = \lambda\tau, \qquad a = (\lambda_0 c)^{\theta}, \qquad \rho = \lambda/\lambda_0 = Q(m)\left(\frac{L}{L_0}\right)^d. \quad (9)$$

The average seismicity rate $\lambda$ is given by $\lambda = \frac{\omega}{1-n}$, which renormalizes the average rate $\omega$ of spontaneous sources by taking into account earthquakes of all generations triggered by a given source: $\lambda = \omega + n\omega + n^2\omega + \ldots$. Because of the assumed statistical independence between event magnitudes, the proportion between spontaneous observable events and their observable aftershocks does not depend on the magnitude threshold and the above expression for the average seismic rate holds also for observable events at different magnitude thresholds of completeness. Finally, $\lambda_0$ is the average seismic rate within a spatial domain $\mathcal{S}_0$ of reference with linear size $L_0$, and $\rho$ takes into account the dependence on the magnitude threshold $m$ for observable events and on the scale $L$ of the spatial domain $\mathcal{S}$ used in the analysis. The first term $(1-n)x$ in the exponential of (8) describes the exponential decreasing probability of having no events as $\tau$ increases due to the spontaneous occurrence of sources. The other term proportional to $x^{1-\theta}$ takes into account the influence through Omori's law of earthquakes that happened before the time window.

*Statistics of recurrence times.*—Consider a sequence of times $\{t_i\}$ of observable earthquakes, occurring inside a given seismic area $\mathcal{S}$. The interevent times are by definition $\tau_i = t_i - t_{i-1}$. The whole justification for the calculation of $P_s(\tau)$ lies in the well-known fact in the theory of point processes [19] that the PDF $H(\tau)$ of recurrence times $\tau_i$ is given by the exact relation

$$H(\tau) = \frac{1}{\lambda}\frac{d^2P_s(\tau)}{d\tau^2}. \quad (10)$$

Substituting (8) in this expression yields our main theoretical prediction for the PDF of recurrence times, which is found to take the form (1) with

$$f(x) = (an\theta\rho^{\theta}x^{-1-\theta} + [1 - n + na\rho^{\theta}x^{-\theta}]^2) \times \exp\left(-(1-n)x - \frac{na\rho^{\theta}}{1-\theta}x^{1-\theta}\right). \quad (11)$$

Expression (11) has a simple physical meaning deduced from (8). The first term $e^{-(1-n)x}$ in (8) controls the exponential decaying probability that there are no earthquakes, given the possible occurrence of observable spontaneous events within the time window $[t, t+\tau]$. The second term $\exp[-Cx^{1-\theta}]$ results from a double time integration of the Omori law, one coming from counting the aftershocks of a given source and one from counting all the possible sources, within the intervals $[t, t+\tau]$ and $[-\infty, t]$, respectively.

While our theoretical derivation justifies the scaling relation (1) observed empirically [9,10], the scaling function $f(x)$ given by (11) is predicted to depend on the criticality parameter $n$, the Omori law exponent $\theta$, the detection threshold magnitude $m$ and the size $L$ of the spatial domain $\mathcal{S}$ under study. While $\theta$ might perhaps be argued to be universal, this is less clear for $n$, which could depend on the regional tectonic context. The situation seems much worse for universality with respect to the two other parameters $m$ and $L$ which are catalog specific. It thus seems that our prediction cannot agree with the finding that $f(x)$ is reasonably universal over different regions of the world as well as for worldwide catalogs [9,10].

It turns out that the dependence on the idiosyncratic catalog-dependent parameters $m$ and $L$ is basically irrelevant as long as $\theta$ is small and $n$ in the range 0.7–1 previously found to be consistent with several other statistical properties of seismicity [17,20]. Note that the condition that $\theta$ be small is fully compatible with many empirical studies in the literature for the Omori law reporting an observable (renormalized) Omori law decay $\sim 1/t^{0.9-1}$ corresponding to $\theta = 0$–$0.1$ [16]. Figure 2 shows the changes of $f(x)$ when varying the magnitude threshold from 0 to 3. These changes of $f(x)$ seem to be within the inherent statistical uncertainties observed in empirical studies [9,10]. The technical origin of the robustness lies in the fact that, for $\theta = 0.03$ say, changing $m - m_0$ from 0 to 6 amounts to changing $\rho$ from 1 ($m = m_0$) to $\rho = 10^{-6}$ ($m = m_0 + 6$), which changes $\rho^{\theta}$ from 1 to only $\rho^{\theta} \simeq 0.66$. We conclude that our theory provides an explanation for both the scaling ansatz (1) and its apparent universal scaling function.

We can squeeze more out of (11) to rationalize the empirical power laws reported by Corral. In particular,

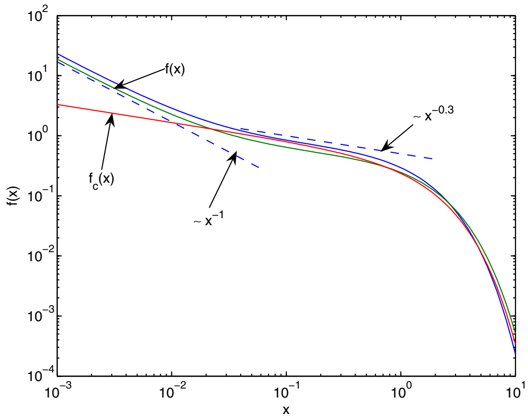


FIG. 2 (color online). Scaling function $f(x)$ defined in (11) for $n = 0.8$, $\theta = 0.03$, $a = 0.76$ and for two values of $m - m_0 = 0, 3$ corresponding to a 1000-fold variation of $\rho = 1; 10^{-3}$. In these synthetic examples, we assume that the spatial domain $\mathcal{S}_0$ corresponds to an average seismicity rate $\lambda_0 \simeq 1$ per day, that the characteristic time scale of the Omori law is $c \simeq 10$ sec., so that $\lambda_c \simeq 10^{-4}$. Then, for $\theta \simeq 0.03$, we have $a \simeq 0.76$. The obtained function $f(x)$ is compared with Corral's empirical fitting function $f_c(x)$ defined in (12) with $g = 0.7$, $\delta = 1.05$, $d = 1.7$, and $C = 0.78$. The dashed lines are the power laws $\sim x^{-0.3}$ and $x^{-1}$.

Corral proposed the following empirical form for $f(x)$ which, in our notations, reads

$$f_c(x) = \frac{C\delta}{d\Gamma(\gamma/\delta)}\left(\frac{x}{d}\right)^{g-1} e^{-(x/d)^\delta}, \quad (12)$$

where $g = 0.67 \pm 0.05$, $\delta = 1.05 \pm 0.05$, $d = 1.64 \pm 0.15$ and $C$ ensures normalization [9,10]. Figure 2 shows indeed that expression (12) with Corral's reported parameter values for $g$, $\delta$, and $d$ fits (11) remarkably well quantitatively. In other words, our theory "explains" Corral's formula by proposing a rigorous derivation of expression (11) which is for all practical purpose, given the noise of the data, identical to Corral's formula (12) for scaled recurrence times larger than about $10^{-2}$. For instance, while the intermediate asymptotics $f(x) \sim x^{\gamma-1} \simeq x^{-0.3}$ proposed by Corral is absent from our theoretical expression (11), it can actually be seen as a long crossover between the power and exponential factors in (11), as shown by one of the dashed lines in Fig. 2.

Interestingly, expressions (11) and (12) depart from each other for $x \lesssim 0.01$. Our theoretical distribution $f(x)$ has the power law asymptotic $f(x) \sim x^{-1}$, which is a direct consequence of Omori's law described explicitly by the first power law factor in front of the exponential in (11). It is absent from expression (12). However, as shown in Fig. 1, its presence is clear qualitatively in real data for the five out of six data sets extracted from [10] on which we have superimposed our theoretical prediction (11). Note that the asymptotic $f(x) \sim x^{-1}$ is also supported from Bak *et al.*'s law $x\varphi(x) \simeq$ const for very small $x$ for the scaled PDF $\varphi(x)$ of multiple regions [8]. However, expression (11) exhibits a visible departure from the data for small scaled recurrence times $x$'s [defined in (9)], which could be attributed to two factors. (i) The linearization of (7) amounts to neglecting the renormalization of the Omori law by the cascade of triggered aftershocks [16]. Taking into account this renormalization effect by the higher-order terms in the expansion (7) improves partially the fit to the data shown in Fig. 1. (ii) It is well known that seismic catalogs are incomplete at short recurrence times [3,21]: the fact that the data are below our theoretical curves is exactly what one should expect given this time incompleteness since the empirical PDF is expected to undersample the number of recurrences times as short times.

Finally, our detailed study shows that comparing (11) with data provides constraints on the parameter $n$: the data definitely exclude small values of $n$ and seem best compatible with $n = 0.7$–1, in agreement with previous constraints [17] suggesting that earthquake triggering is a dominant process.

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