Difference between a Photon's Momentum and an Atom's Recoil

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When an atom absorbs a photon from a laser beam that is not an infinite plane wave, the atom's recoil is less than $\hbar k$ in the propagation direction. We show that the recoils in the transverse directions produce a lensing of the atomic wave functions, which leads to a frequency shift that is not discrete but varies linearly with the field amplitude and strongly depends on the atomic state detection. The same lensing effect is also important for microwave atomic clocks. The frequency shifts are of the order of the naive recoil shift for the transverse wave vector of the photons.

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It is well known that an isolated atom recoils with a momentum $\hbar k$ when it absorbs or emits a photon of wave vector k from an infinite plane wave. A laser beam that propagates in the z direction and is not infinite in the transverse directions must have nonzero transverse wave vectors k_x and k_y. Therefore, for physical laser beams, k_z must be less than $k = (k_x{}^2 + k_y{}^2 + k_z{}^2)^{1/2}$, raising the question of whether the photon recoil of an atom is less than $\hbar k$. Three conceptually appealing choices are as follows: (1) the photon recoil is $\hbar k$ in the z direction; (2) the photon recoil is only $\hbar k_z$ in the z direction; (3) the photon recoil is $\hbar k_z$ in the z direction, and $\pm\hbar k_x$ and $\pm\hbar k_y$ in the x and y directions, so that the magnitude of the recoil is $\hbar k$. Recent analysis of precision measurements of the photon recoil [1–3], which helps determine the fine structure constant, has shown that the first choice is excluded [4]—the recoil in the z direction must be $\hbar k_z$ because the recoil is intimately connected to the gradient of the phase of the field [5]. Here we analyze the transverse recoil and the systematic frequency shift that it produces when cold atoms interact with electromagnetic fields that have transverse sizes of the order of 1 cm. These transverse wave vectors are in the microwave regime and produce effects that need to be considered at the level of the present accuracy of atom interferometer experiments and microwave atomic clocks.

None of the three choices above is fully correct. The atoms in these experiments are localized to typically less than 1 cm, spanning less than one transverse wavelength of the field. Therefore the atoms do not diffract with discrete microwave photon recoils. Instead, they experience deflections in the transverse directions that are determined by the local gradient of the field strength. The field also acts as both a positive lens and a negative lens on the atomic wave functions. These effects, in the microwave Stern-Gerlach regime, are analogous to those in the optical Stern-Gerlach regime, in which the localization is less than an optical wavelength [6]. We show that the deflection and the lensing of the wave functions produce novel frequency shifts for atomic clocks and interferometers.

Precise measurements of the photon recoil with atom interferometers [1,3] and precision spectroscopy [2] have used laser beam waists of 2 mm and 1 cm, for which the difference between $\hbar k_z$ and $\hbar k$ gives 8 to 0.4 ppb corrections. Current experiments have an accuracy of 4–6 ppb and future work aspires to 0.1 ppb accuracy [1].

Laser-cooled microwave atomic clocks have advanced to the point where the frequency shift due to microwave photon recoils must be considered [7,8]. When an atom at rest absorbs a photon from an infinite plane wave, conservation of energy and momentum for the recoiling atom leads to a shift of the resonance frequency of $h\Delta\nu = \hbar^2 k^2/2$ m [9]. For the cesium clock transition, the recoil velocity is 0.1 μm/s and, although very small, it naively leads to a frequency shift of 1.5×10^{-16} [10], which is comparable to the current accuracy of 5×10^{-16} for laser-cooled cesium fountain clocks [8]. We show that instead of the usual recoil shift $\Delta\nu$, a comparable frequency shift occurs in clocks due to lensing and deflection of the atoms that scales in novel ways with the interrogation time and the size of the wave functions.

In clocks and atom interferometers, the atoms pass through two or more oscillatory fields that are separated by interrogation times T, as depicted in Fig. 1 for a clock [8,11]. For a resonant field, the Hamiltonian for the internal degrees of freedom is $H_{\rm int} = \frac{1}{2}\hbar\omega\sigma_3 + \frac{1}{2}\hbar\Omega(\mathbf{r}, t) \times \cos(\omega t)\sigma_1$. Here $\Omega(\mathbf{r}, t)$ is the Rabi frequency [12], ω is the transition frequency, and $\boldsymbol{\sigma}$ is the usual Pauli matrices. We make the rotating wave approximation and rotate to a basis in which $H_{\rm int}$ is diagonal to get the usual dressed states $|1\rangle$ and $|2\rangle$. The Schrödinger equation is then $i\partial_t\Psi = [-\hbar\nabla^2/2m + \frac{1}{2}\Omega(\mathbf{r}, t)\sigma_3]\Psi$. For atom interferometers, the dressed states for a traveling wave are equal superpositions of the ground and excited states, $|g, k_{0z}\rangle$ and $|e, k_{0z} + k_{1z}\rangle$, which include the photon recoil $\hbar k_{1z}$ in the z direction. Note that we do not include the photon sector in our dressed states. The transverse forces are described by this Schrödinger equation.

We solve the Schrödinger equation using a Gaussian wave packet for Ψ that has an initial central position $\mathbf{r_0}$ and momentum $\mathbf{k_0}$, and a position width $\boldsymbol{\Delta}$. The lowest

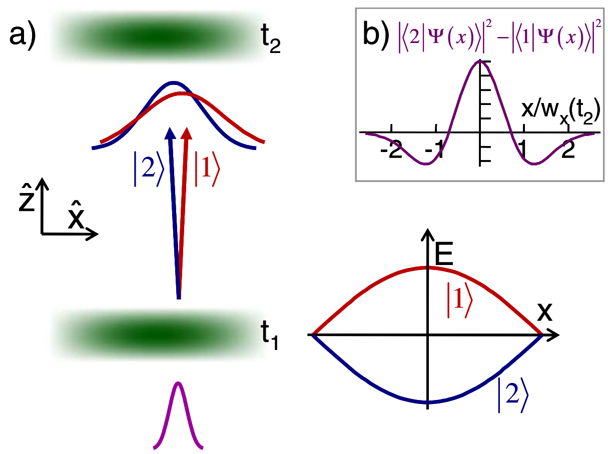

FIG. 1 (color online). (a) An atom interacts with an electromagnetic field for a short time at t_1 and t_2. The spatial variation of the dipole energy for the two dressed states, depicted in the graph, deflects the atomic wave functions and acts as a positive lens on the dressed state $|2\rangle$ and a negative lens on $|1\rangle$. (b) The spatially dependent difference of dressed state populations leads to a frequency shift of the order of the recoil shift for the transverse wave vector.

order effect of the field interaction at time t_1 is given by the Raman-Nath approximation, which neglects kinetic energy terms in the Schrödinger equation. Integrating the Schrödinger equation with respect to time gives a position dependent phase shift to the wave function. For an atomic clock with a rectangular microwave cavity, the field is $B(\mathbf{r}) = \cos(k_{1x}x)\cos(k_{1z}z)$ with wave vector $\mathbf{k_1}$ and $k_{1y} = 0$. It gives a scattering phase shift of

$$U(t^+, t^-) = \exp[-\tfrac{1}{2}i\varphi(k_z)\cos(k_{1x}x)\sigma_3] \tag{1}$$

acting on $\Psi(x, y, k_z, t^-)$. The natural basis for this interaction is the transverse position and longitudinal momentum k_z, instead of a pure position basis. Here $t^{\mp}$ are just before and after the field acts for a short time at t_1, $\phi(k_z) = \phi_0 k_{0z}/k_z$, and $\phi_0 = \pi/2$ for a $\pi/2$ pulse. If the atomic wave functions spanned many transverse wavelengths, it would be helpful to expand $\exp[\pm\frac{1}{2}i\phi(k_z)\cos(k_{1x}x)]$ as a sum of Bessel functions $J_n[\frac{1}{2}\phi(k_{1x})]\exp[\pm \mathrm{i}nk_{1x}x]$ since each term would represent the scattering of n photons. Here, because the atoms are restricted to less than a transverse wavelength, this expansion leads to many interfering terms [10] and little insight. Furthermore, it makes analytic calculations intractable and numerical calculations difficult, especially when both transverse dimensions are important [10], as in most interferometers and clocks.

Here, we expand $\cos(k_{1x}x)$ in Eq. (1) as $1 - \frac{1}{2}{k_{1x}}^2x^2$. This gives a quadratic phase variation and allows us to treat the problem analytically, even when both transverse dimensions are important [13]. A quadratic phase variation is a general feature. For interferometers, the field strength of Gaussian laser beams varies quadratically around $r = 0$. For clocks, the fields of the rectangular cavity above and the commonly used cylindrical cavity [8] both vary quadratically around $x = y = 0$.

As in classical optics, the quadratic phase variation acts as a lens on the incident wave. The lensing effect in Fig. 1 is clear as the atom propagates to a second field interaction at $t_2 = t_1 + T$.

$$|\langle 2|\Psi(x, t_2)\rangle|^2 = \frac{e^{-(x-x_2+\varepsilon_x)^2/(w_{2x}-\delta_x)^2}}{2\sqrt{\pi}(w_{2x} - \delta_x)}. \tag{2}$$

Here $w_x(t) = [\hbar^2t^2/2m^2{\Delta_x}^2 + 2{\Delta_x}^2]^{1/2}$ is the width and x_2 is the center of the wave packet in the absence of the field, $w_{1x} = w_x(t_1)$, $w_{2x} = w_x(t_2)$, $\delta_x = 2\pi T\phi(k_z)\nu_R w_{1x}$ describes the focusing of the wave packet, and $\epsilon_x = \delta_x x_1/w_{1x}$ is the deflection of the wave function due to the atom traversing the first interaction zone a distance x_1 off axis. The usual recoil shift that is associated with k_{1x} is $\nu_R = \hbar {k_{1x}}^2/4\pi m$, and we have neglected terms of order $m{k_{1x}}^2{\Delta_x}^4/\hbar T$. The form of $|\langle 1|\Psi(x, t_2)\rangle|^2$ is Eq. (2) with the signs of ϵ_x and δ_x reversed. The dipole energy variation [Fig. 1(a)] produces a spatially dependent force that acts as a positive lens for the $|2\rangle$ state and a negative lens for the $|1\rangle$ state. In atom interferometers, there are additional pulses before the final field interaction and they further contribute to the lensing of the wave function. Typical parameters for Cs atoms at 1 μK are $w_x(t_1) = 1$ mm, $x_1 = 2$ mm, $k_{1x} = 100\ \mathrm{m}^{-1}$, and $T = 0.5$ s. These give $\delta_x = 2$ nm and $\epsilon_x = 4$ nm, which are much less than w_{2x}.

We now calculate the frequency shift due to the transverse momentum changes. For the last field interaction, the temporal phase of the field is shifted by $\Phi = \pm\pi/2$, which is a phase shift of the dressed states of $(\mathbf{1} \mp i\sigma_1)/2^{1/2}$. We apply Eq. (1) to Ψ with wave vector $\mathbf{k_2}$, where normally $\mathbf{k_2} = \mathbf{k_1}$. The resulting difference of the excited state populations for $\Phi = \pm\pi/2$ is proportional to the frequency shift. In terms of the dressed states, this is

$$\delta P = [|\langle 2|\Psi\rangle|^2 - |\langle 1|\Psi\rangle|^2]\sin[\varphi(k_z)\cos(k_{2x}x)]. \tag{3}$$

Here the probability densities for the dressed states are given by Eq. (2). This is divided by the amplitude of the Ramsey fringes, ΔP_R, to get the frequency shift $\delta\nu = \delta P/2\pi T\Delta P_R$. We calculate ΔP_R by taking the difference of the excited state probability for $\Phi = 0$ and $\Phi = \pi$:

$$\Delta P_R = 2\,\mathrm{Im}[\langle\Psi|1\rangle\langle 2|\Psi\rangle]\sin[\varphi(k_z)\cos(k_{2x}x)].$$

Equation (3) shows that there will be a frequency shift whenever one of the dressed states has a greater detection probability. The lensing will generally give the $|2\rangle$ state a greater detection probability at the center [Fig. 1(b)]. Expanding Eq. (3) to first order in δ_x and ϵ_x,

$$\delta P = \frac{e^{-(x-x_2)^2/{w_{2x}}^2}}{\sqrt{\pi}w_{2x}}\left\{\frac{\delta_x}{w_{2x}}\left[1 - 2\left(\frac{x - x_2}{w_{2x}}\right)^2\right] - 2\varepsilon_x\frac{x - x_2}{w_{2x}^2}\right\} \times \sin[\varphi_0\cos(k_{2x}x)]. \tag{4}$$

Here we take $k_{2y} = 0$ for clarity and also neglect the variation of $\phi(k_z)$ because the spread of k_z is typically of order 1% of k_{0z}. A simple case to consider is detecting only a small region around $x = 0$ of a wave packet that propagates on the axis ($x_1 = x_2 = 0$). For a $\pi/2$ pulse, $\phi_0 = \pi/2$ and the $\sin[\cdots]$ term in Eqs. (3) and (4) is essentially equal to 1. This gives $\delta\nu = \phi_0 \nu_R w_{1x}/w_{2x}$. Therefore, the frequency shift due to the lensing is of the order of the usual recoil shift, but it increases linearly with the field amplitude (ϕ_0) and also depends on the size of the wave functions at the field interactions. The ratio of wave function widths w_{1x}/w_{2x} is essentially equal to $t_1/(t_1 + T)$, where T/t_1 typically ranges from 1 to 5. For $T/t_1 = 3.7$ and two $\pi/2$ pulses, $\delta\nu \approx \nu_R/3$.

In addition to the frequency shift due to focusing, the deflection of the wave function may also lead to a frequency shift. It is described by the term proportional to ϵ_x in Eq. (4). Again, if we detect a small region around $x = 0$, the frequency shift due to the deflection of the wave packet is $\delta\nu = \phi_0 \nu_R x_1 x_2 / {w_{2x}}^2$. This averages to zero if x_1 and x_2 are uncorrelated, e.g., for a large high-temperature source. We therefore average Eq. (4) over the source distribution and include the effects of the state detection.

The state detection process plays a critical role in determining the magnitude of the lensing frequency shift. The frequency shift from Eq. (3) is very small if the atomic cloud is uniformly detected because each atom is equally split between both dressed states and usually $\sin[\cdots]$ can be neglected. However, several effects inhibit uniform detection. These include finite laser beams, the fluorescence collection optics, and apertures in the experiment. In Fig. 1(b) we plot the spatial variation of the focusing frequency shift, Eq. (4) with $\epsilon_x = 0$. For there to be a large cancellation of the lensing shift, the detection intensity must be uniform over the entire $1/e^2$ diameter of the atomic cloud. If the excited state is detected with a laser beam that has the same width as the final atomic cloud, the frequency shift is half as large as that for detecting only the central region. In atomic clocks, the most spatially selective element is the aperture of the microwave cavity. It has a typical radius of 5 mm, whereas the final atomic cloud has a $1/e^2$ radius of 1 cm [8].

Next we average the lensing frequency shift over the initial spatial and velocity distributions, and also study the effects of the apertures. The result is independent of the coherence length of the atomic wave functions [14] because, in essence, the frequency shift from Eq. (3) is proportional to the difference of the dressed state populations, whether the dressed state populations are for one atom or the ensemble. Neglecting the variation of $\sin[\cdots]$ in Eq. (4), we average over the initial position and velocity distributions that are centered on the cavity axis, with $|x_1| < a$, where a is the aperture radius and the $1/e$ velocity width is $u = (2k_B T_{\rm th}/m)^{1/2}$. Integrating over position $|x| < a$ at time t_2, we get

$$\frac{\delta\nu}{\nu_R} = \varphi_0 \frac{\frac{2}{\sqrt{\pi}} \frac{w_{c1x}}{u(t_2-t_1)w_{0x}} \int_{-a}^{a} x_1 e^{-[{x_1}^2 {w_{c2x}}^2 - 2a x_1 ({w_{0x}}^2 + u^2 t_1 t_2) + a^2 {w_{c1x}}^2]/u^2 (t_2-t_1)^2 {w_{0x}}^2} dx_1}{\sin(\varphi_0) \int_{-a}^{a} e^{-x_1^2/{w_{c1x}}^2} Erf[\frac{x_1({w_{0x}}^2 + u^2 t_1 t_2) + a {w_{c1x}}^2}{u(t_2-t_1) w_{0x} w_{c1x}}] dx_1}. \tag{5}$$

Here, the size of the cloud at t_1 and t_2 are $w_{c1x} = ({w_{0x}}^2 + u^2 {t_1}^2)^{1/2}$ and $w_{c2x} = ({w_{0x}}^2 + u^2 {t_2}^2)^{1/2}$. The integral limits represent the constraint of the aperture at t_1 and we have neglected diffraction. Often w_{c1x} is less than a, and then the integrals are essentially over all x_1 (no aperture at t_1), giving

$$\frac{\delta\nu}{\nu_R} = \varphi_0 \frac{\frac{2a}{w_{c2x}} [\frac{t_1}{t_2} + \frac{{w_{0x}}^2}{{w_{c2x}}^2}(1 - \frac{t_1}{t_2})] e^{-a^2/{w_{c2x}}^2}}{\sqrt{\pi} \sin(\varphi_0) Erf(a/w_{c2x})}. \tag{6}$$

In Fig. 2 we show the lensing frequency shift from Eqs. (5) and (6) as a function of a and w_{0x}. For very cold (1 nK) atoms, the thermal expansion of the cloud is small. If $w_{0x} \ll a$, the atoms are uniformly detected and so there is a negligible transverse recoil shift. For a typical temperature of 1 μK, the cloud spreads to fill the second aperture, and when $w_{0x} \to 0$, a larger aperture gives more uniform detection and thus a smaller frequency shift. For a small (1 mm) aperture, the deflection frequency shift becomes smaller as the initial size w_{0x} increases because x_1 is less correlated with x_2. For $a = 5$ mm, 1 μK is not hot enough for x_1 and x_2 to be uncorrelated, so the more important effect is a larger average dipole force as w_{0x} increases to 5 mm. In Fig. 2(b), the solid curves first rise due to larger dipole forces, and then fall with more uniform detection. If the first aperture is removed [dashed curves, Eq. (6)], the atoms see larger dipole forces if $w_{c1x} > a$, yielding a larger frequency shift.

In contrast to the usual recoil shift, the size of the atom cloud at the first field interaction affects the lensing frequency shift. In Fig. 3, the solid lines show the variation of the lensing frequency shift on the time t_1 between the end of the laser cooling and the first field interaction. For a small initial cloud size, $w_{0x} = 1$ mm, the shift is very small if the atom immediately interacts with the field [Eq. (6)]. If t_1 is long, then the effect of the initial size is relatively small. For an initial size of $w_{0x} = 5$ mm, the frequency shift is on the order of ν_R, even for $t_1 \to 0$. In an operating fountain clock, t_1 can be increased by decreasing the launch velocity. For large t_1, the interrogation time is dramatically shorter, and eventually $t_2 \to t_1$. The dashed lines correspond to interrogation times $T = 1$ to 0.1 s.

The lensing frequency shift also depends upon the initial atomic state and the field amplitude. If the atom is initially prepared in the upper state $|e\rangle$, the difference in the population of $|e\rangle$ after detection from Eq. (3) is unchanged because it depends only on the difference of the dressed state populations. However, the sign of the Ramsey fringes reverses and therefore the frequency shift reverses if the initial state is changed. As noted above, the lensing fre-

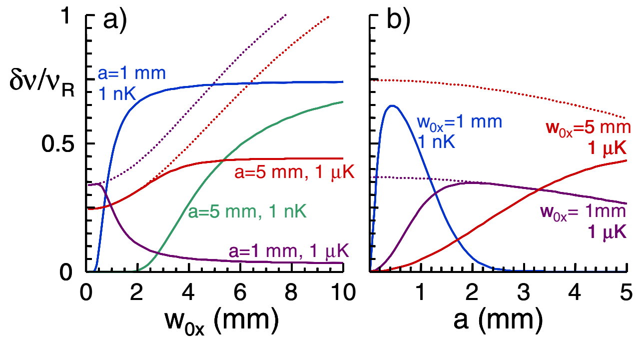

FIG. 2 (color online). Lensing frequency shift as a fraction of the transverse photon recoil frequency shift ν_R vs (a) the initial size of the atomic cloud w_{0x} and (b) the aperture radius a for Cs atoms at 1 μK and 1 nK, with $t_1/t_2 = 0.22$. The dotted lines represent no aperture at time t_1 [Eq. (6)] and the solid lines apertures at t_1 and t_2 [Eq. (5)]. The text describes three effects manifested in the curves: detection uniformity, the variation of the average dipole force with cloud and aperture size, and the ensemble average as a function cloud size and temperature. Generally, the lensing frequency shift is of the order of the transverse photon recoil frequency shift.

quency shift is linear in the field amplitude, given by ϕ_0, around $\phi_0 = \pi/2$. At higher amplitudes, the shift will continue to increase almost linearly for $\phi_0 = 3\pi/2$, $5\pi/2$, $7\pi/2$, etc., as shown by Eqs. (5) and (6) [15]. For higher powers, the ${k_{1x}}^4 x^4$ terms in Eq. (1) and $\sin[\phi_0 \cos(k_{2x}x_2)]$ in Eq. (3) may be significant and can be calculated.

The spatial extent and sensitivity of atom interferometers can be increased by inserting additional photon recoils, as many as 140, in the sequence of pulses [1,3,16]. These fields also produce a lensing of the wave functions with fractional errors of order ${k_{1x}}^2/{k_{1z}}^2$. However, the experiments are often performed as dual interferometers, and both interferometers will nominally have the same frequency offset. Thus, when the result is the difference frequency, it is less sensitive to the transverse momentum changes.

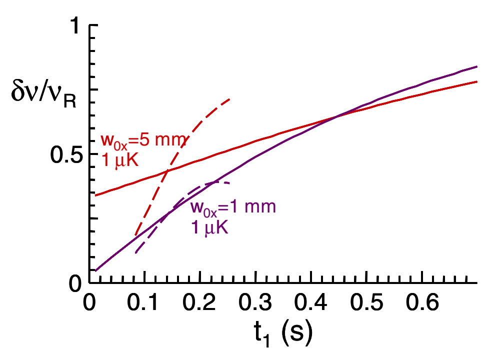

FIG. 3 (color online). Lensing frequency shift for Cs atoms at 1 μK as a function of t_1, the time between the release of the cloud of atoms and the first field interaction. For the solid lines, t_1 is changed by varying the distance to the first field interaction with a fixed interrogation time of $T = t_2 - t_1 = 0.5$ s. For the dashed lines, the distance is fixed and the atomic-fountain launch velocity is varied, which changes both t_1 and T. Here, $t_1 = 0.14$ s for $T = 0.5$ s, and T ranges from 1 to 0.1 s.

In summary, when an atom absorbs a photon from a laser beam propagating in the z direction, it experiences a discrete recoil in the z direction that is less than $\hbar k$ if the laser beam is not infinite in the transverse directions. The spatial gradient of the dipole energy in the transverse directions produces transverse forces. These lead to a small lensing of the atomic wave function, not to resolved diffraction with discrete momenta. The fields of microwave cavities produce similar dipole forces and the same lensing effects. If the final state detection is uniform, there is essentially no frequency shift associated with the transverse dipole forces and the recoil shift is given by only $\hbar k_z$. In general, the transverse dipole forces do not affect the average position or width of the atomic wave function, but they do affect the phase in a way that can be expressed as a difference of dressed state populations. When the central part of the atomic distribution is preferentially detected, there is a frequency shift of the order of the recoil shift for the transverse wave vector. For a traveling wave laser beam, the total recoil shift is then closer to the usual recoil shift given by $\hbar k$. The shift due to the transverse dipole forces reverses with the initial atomic state and depends on the field amplitude, the state detection homogeneity, and the size of the atomic cloud at the field interactions.

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[1] A. Wicht *et al.*, Phys. Scr. **T102**, 82 (2002).
[2] P. Cladé *et al.*, Phys. Rev. Lett. **96**, 033001 (2006).
[3] D. S. Weiss, B. C. Young, and S. Chu, Phys. Rev. Lett. **70**, 2706 (1993).
[4] A. Wicht *et al.*, Phys. Rev. A **72**, 023602 (2005).
[5] Campbell *et al.* [Phys. Rev. Lett. **94**, 170403 (2005)] show that the index of refraction also affects the recoil.
[6] T. Sleator *et al.*, Phys. Rev. Lett. **68**, 1996 (1992).
[7] C. Vian *et al.*, IEEE Trans. Instrum. Meas. **54**, 833 (2005).
[8] R. Wynands and S. Weyers, Metrologia **42**, S64 (2005).
[9] J. L. Hall, C. J. Bordé, and K. Uehara, Phys. Rev. Lett. **37**, 1339 (1976).
[10] P. Wolf and Ch. Borde, quant-ph/0403194.
[11] The same effects occur for a single Rabi pulse.
[12] For microwave clocks, $\hbar\Omega = -\boldsymbol{\mu} \cdot \mathbf{B}$, where $\boldsymbol{\mu}$ is the magnetic dipole moment. For atom interferometers, $\hbar\Omega = -\mathbf{d} \cdot \mathbf{E}$, where $\mathbf{d}$ is the electric dipole moment.
[13] The recoil shifts from two dimensions simply add.
[14] P. D. Featonby *et al.*, Phys. Rev. Lett. **81**, 495 (1998).
[15] Equations (5) and (6) are trivially singular for $\phi_0 = \pi$, 2π, 3π, etc. For these ϕ_0, $\Delta P_R = 0$ and also $\delta P = 0$.
[16] J. M. McGuirk, M. J. Snadden, and M. A. Kasevich, Phys. Rev. Lett. **85**, 4498 (2000).