

Spectrum of the Andreev Billiard and Giant Fluctuations of the Ehrenfest Time

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The density of states in the semiclassical Andreev billiard is theoretically studied and shown to be determined by the fluctuations of the classical Lyapunov exponent λ . The rare trajectories with a small value of λ give rise to an anomalous increase of the Ehrenfest time $\tau_E \approx |\ln \hbar|/\lambda$ and, consequently, to the appearance of Andreev levels with small excitation energy. The gap in spectrum is obtained, and fluctuations of the value of the gap due to different positions of superconducting lead are considered.

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Introduction.—The density of states in a metallic island coupled to a superconductor is modified due to the proximity effect [1]. The changes are most pronounced in the vicinity of Fermi energy, where there opens a gap in the spectrum of excitations. A ballistic chaotic normal region coupled to a superconductor via the small constriction (NS interface) is called the Andreev billiard [2]. The spectrum of such billiards was calculated a decade ago [3] assuming the random matrix description of the quantum dynamics. This approximation is valid if the number of channels N supported by the NS interface is small. In spite of large efforts [4–11], there is currently no reliable calculation of the spectrum in the most interesting case of the semiclassical Andreev billiard, when the number of open superconducting channels scales with the Planck constant as $N \sim 1/\hbar$.

The properties of Andreev billiards are governed by the Andreev reflection, when an electron trajectory is retraced by the hole that is produced upon absorption of a Cooper pair at the NS interface. At the Fermi energy E_F , the classical dynamics of the hole is the time reverse of the electron dynamics, so that the motion is strictly periodic. This periodicity, however, comes in conflict with a quantum-mechanical evolution, since each Andreev reflection adds an extra phase $\pi/2$ to the electron-hole wave function. This phase is compensated by the difference of classical actions of electron ($E_F + \varepsilon$) and hole ($E_F - \varepsilon$) along the trajectory, where ε is the excitation energy. The longer the interval t is between Andreev reflections, the smaller energy suffices to produce the missing phase $\varepsilon = \hbar\pi/2t$. On the other hand, the probability of a particle trajectory to not touch the NS interface for a long time becomes exponentially small $\sim e^{-t/t_D}$, with t_D being a dwell time. This leads to the prediction of an exponential suppression of the density of states [3,12]

$$\rho(\varepsilon) \approx N(\hbar\pi/\varepsilon^2 t_D) \exp(-\hbar\pi/2\varepsilon t_D). \quad (1)$$

However, the density of states at small excitation energies, corresponding to large times, is not captured by this formula. A new time scale responsible for the spectrum at low energies [4] is the Ehrenfest time $\tau_E \sim |\ln \hbar|/\lambda$, where λ is

a Lyapunov exponent in the normal billiard. For longer times, the initial quantum wave packet $\Delta x \Delta p \sim \hbar$ acquires a macroscopic size due to exponential $\sim e^{\lambda t}$ divergency of trajectories. This invalidates the trajectory based derivation that led to (1). It is believed that, below a certain energy $\varepsilon_{\text{gap}} \sim \hbar/\tau_E$, the density of Andreev states vanishes exactly, $\rho(\varepsilon < \varepsilon_{\text{gap}}) \equiv 0$, but the magnitude of the gap and the mechanism of its formation remain a subject of controversial discussion [6,7,10].

At finite times, the value of the Lyapunov exponent λ depends on the specific trajectory [13], leading to fluctuations of the Ehrenfest time. Both the Andreev spectrum [3–11] and the quantum to classical crossover in ballistic transport [14–18] attracted a great deal of interest recently, but the role of fluctuations of the Ehrenfest time was never investigated. In this Letter, we show how the low energy density of states is determined by the large Ehrenfest time fluctuations and solve the long-standing problem of the Andreev gap.

The distribution of finite time Lyapunov exponents is parametrized as (here τ_0 is, e.g., an averaged time between bounces at the walls of normal billiard)

$$P(\lambda, t) = \tau_0 \exp[tF_t(\lambda)]. \quad (2)$$

In the case of chaos, the limit $F_{t \gg \tau_0}(\lambda) = F(\lambda)$ exists, with $F(\lambda)$ specific for the dynamical model [19]. The function $F(\lambda)$ has a maximum at $\lambda = \lambda_0$, which is the conventional self-averaging Lyapunov exponent [$F(\lambda_0) = 0$]. Since all small values of $\lambda > 0$ are present in the distribution (2), one may always find (rare) trajectories with any large value of the Ehrenfest time. To build a semiclassical eigenfunction of a two-dimensional billiard, however, one needs a family of trajectories, all having the same interval t between the Andreev reflections. The explicit construction of discrete Andreev levels from the tube of trajectories (whose transverse Poincaré section is quantized via the Bohr-Sommerfeld rule) is presented in Ref. [6]. The gap in the spectrum is determined by the largest time for which the number of trajectories is enough to form at least one eigenstate. Quantitative counting of the number of trajectories is done with the use of the concept of a transmission

band [15,20]. This leads us to the expressions for the gap and the density of states [(13) and (16) below] depending on $F(\lambda)$, t_D , and a number N of open channels in the NS interface.

Stroboscopic model.—All of the essential features of a generic Andreev billiard are captured by the stroboscopic model of the Andreev billiard, which was developed in Ref. [8] in order to use the advantages of an open kicked rotator [21] for large scale quantum-mechanical simulations. Here we present a semiclassical solution of this model. First, the quantum kicked rotator, a counterpart of the classical standard map [22],

$$p_{n+1} = p_n + \frac{KI_0}{\tau_0} \sin\theta_n, \quad \theta_{n+1} = \theta_n + \frac{\tau_0}{I_0} p_{n+1}, \quad (3)$$

is defined by the Floquet operator

$$U = \exp\left(i \frac{\hbar\tau_0}{2I_0} \frac{d^2}{d\theta^2}\right) \exp\left(i \frac{KI_0 \cos\theta}{\hbar\tau_0}\right). \quad (4)$$

Here I_0 , τ_0 , and K are the moment of inertia, the interval between kicks, and the kicking strength, respectively. Next, introduce the dimensionless Planck constant $\hbar_{\text{eff}} = \hbar\tau_0/I_0$. If $\hbar_{\text{eff}} = 2\pi/M$ with integer M , the coordinate and the momentum $\hat{p} = -i\hbar_{\text{eff}}d/d\theta$ take discrete values $\theta_k = 2\pi k/M$, $p_m = 2\pi m/M$, $k, m = 1, 2, \dots, M$. The Floquet operator now becomes a $M \times M$ matrix.

The electron and the hole components of the wave function of the Andreev kicked rotator span over the doubled $2M$ -sites Hilbert space with their evolution given by the normal (U) or conjugated (U^*) Floquet operators. The electron is converted into the hole by reflection at the N -channel superconducting lead, attached at $\theta_1 < \theta < \theta_2$ ($\theta_2 - \theta_1 = \hbar N$). This is done with the help of projection matrix Q whose N only nonzero elements are $Q_{k,k} \equiv 1$, for $\theta_1 < \hbar k < \theta_2$. Andreev levels are found from

$$\mathcal{U}\psi = e^{i\varepsilon\tau_0/\hbar}\psi, \quad \mathcal{U} = \begin{pmatrix} (1-Q)U & -iQU^* \\ -iQU & (1-Q)U^* \end{pmatrix}. \quad (5)$$

The classical limit corresponds to $M, N \rightarrow \infty$, while the dwell time $t_D = \tau_0 M/N$ is fixed. A classical particle at any time $t = n\tau_0$ has a definite position either inside the normal region or at the interface. A semiclassical quantization of the map (5) requires a construction of the quantum states having a similar property. A formal description of the wave packet ϕ , which is injected from the superconductor, stays inside the billiard for $n-1$ kicks, and then hits the NS interface, is given by ($0 < m < n$)

$$Q\phi = \phi, \quad QU^m\phi = 0, \quad QU^n\phi = U^n\phi. \quad (6)$$

Provided such a solution is found, one easily builds $2n$ eigenfunctions with the eigenvalues ($r = 0, \dots, n-1$)

$$\varepsilon_{nr} = \pm \hbar\pi \frac{2r+1}{2n\tau_0}, \quad \psi = \begin{pmatrix} \sum_{k=0}^{n-1} U^k e^{-ik\varepsilon\tau_0/\hbar} \phi \\ -\sum_{k=1}^n U^k e^{ik\varepsilon\tau_0/\hbar} \phi \end{pmatrix}. \quad (7)$$

ε_{n0} with the largest possible n constitutes a gap. Below, we always consider only the levels with $r = 0$. Equations (6)

and (7) are the analog of adiabatic quantization, developed in Ref. [6] for the generic Andreev billiard.

Strictly speaking, Eqs. (6) for any n have no solutions. However, for $\hbar \rightarrow 0$ where exists $\approx N$ linearly independent wave packets satisfying (6) with practically any desired accuracy, $\phi^\dagger QU^m \phi \sim e^{-1/\hbar}$. Finding the number of such solutions reduces to the calculation of certain phase-space areas called transmission bands [15].

Transmission bands.—We call the transmission band a simply connected part of the phase-space area of the NS interface, $\theta_1 < \theta < \theta_2$, $0 < p_{\text{eff}} < 2\pi$, each point θ, p of which visits the interface at the n th iteration of the map (and does not visit it earlier). The image of the stripe $\theta_1 < \theta < \theta_2$ after n iterations is another long and narrow (curved) stripe of width $\sim e^{-\lambda t} N/M$ (see examples in Refs. [16,17]). Phase-space overlaps of the NS interface with its image, the transmission bands, are the areas with approximately the shape of a parallelogram whose long and short sides have a length $\sim N/M$ and $\sim e^{-\lambda t} N/M$, respectively. The number N_b of (families of) Andreev levels (7) supported by a single transmission band is calculated as its area divided by $2\pi\hbar_{\text{eff}}$,

$$N_b \approx (N^2/M)e^{-\lambda t}. \quad (8)$$

The total number of levels composed from trajectories having the time $t = n\tau_0$ between hitting the NS interface, whose Lyapunov exponent falls in the interval $d\lambda$, is (2)

$$dN_t = NP(\lambda, t)e^{-t/t_D}(\tau_0/t_D)d\lambda. \quad (9)$$

These levels originate from the many transmission bands of various size: $N_t = \sum N_b$. We may use (8) to express λ through N_b and to find the distribution of levels over the sizes of the bands

$$\frac{dN_t}{d \ln N_b} = \frac{N^2\tau_0}{Mt} \exp\left\{tF_t\left(\frac{1}{t} \ln \frac{N^2}{MN_b}\right) - \frac{t}{t_D}\right\}. \quad (10)$$

The distribution $dN_t/d \ln N_b$, found numerically for the model (5), is shown by the histogram in Fig. 1 for times $t = 6\tau_0$ and $t = 7\tau_0$. The choice of $\hbar_{\text{eff}} = 2\pi \times 10^{-7}$

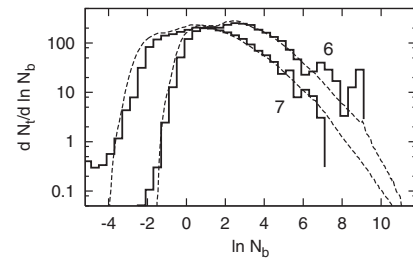


FIG. 1. Distribution of Andreev levels with $\varepsilon = \hbar\pi/12\tau_0$ and $\varepsilon = \hbar\pi/14\tau_0$ over transmission bands of different size, corresponding, respectively, to the 6th and 7th iterations of the standard map averaged over 200 positions of the NS interface. Dwell time $t_D = 10$; kicking strength $K = 10$. The dashed lines show the theoretical prediction (10), with the function F found numerically for 5 and 6 iterations. For $M = 10^7$, the value $\lambda = 0$ would correspond to $N_b \approx 10^5$ in Eq. (8).

introduces a quantum-mechanical scale in classical area counting. Direct quantum-mechanical calculation of energy levels in such an Andreev billiard would require diagonalization of the matrix of the size $M = 10^7$, which is beyond the reach for existing computers.

The dashed lines in Fig. 1 show the predicted distribution [Eq. (10)] with the numerically found function $F_i(\lambda)$ for times $t = 5\tau_0, 6\tau_0$. The number of iterations in the theoretical formula is reduced by 1 because the effective strong ($\sim K$) stretching of the phase-space image of the NS interface starts only from the second iteration.

Even though the trajectories with $\lambda \approx 0$ are present for both times shown in Fig. 1, the largest transmission bands expected from Eq. (8) at the smallest values of the Lyapunov exponent are absent [23]. Also, the area of the largest existing transmission band is smaller for a larger time. This is the actual reason for the formation of the gap in the excitation spectrum.

The gap in spectrum is given (7) by the *longest* time for which there exist the transmission bands with $N_b > 1$

$$\varepsilon_{\text{gap}} = \hbar\pi/2t_{\text{max}}. \quad (11)$$

To illustrate the mechanism of the creation of the gap, we show in Fig. 2 the distribution $dN_i/d\ln N_b$ for times $t/\tau_0 = 2-16$ (shifted vertically). Counting the small areas numerically is a very time-consuming procedure [25]. Starting from the 10th iteration, more than half of the Andreev levels are missing due to the errors in area counting. Still, the results for the largest transmission bands are trustable for all times presented in the figure. The existing estimates of the Andreev gap [6,7,10] neglected the fluctuations of the Lyapunov exponents (shown by narrow peaks connected by a line). The right solid line in Fig. 2 shows the (asymptotic) time dependence of the largest

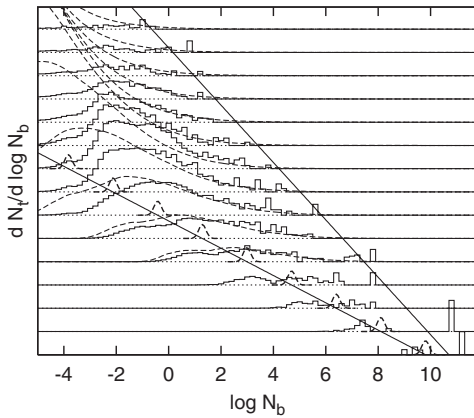


FIG. 2. The number of Andreev levels vs logarithm of the area of transmission band $dN_i/d\ln N_b$ for the stroboscopic model with $K = 11$, $\hbar_{\text{eff}} = 2\pi \times 10^{-7}$. The superconducting lead is attached at $\pi/10 < \theta < 3\pi/10$. Histograms for a different number of kicks $n = 2-16$ are offset vertically and multiplied by 2^n . For $n \geq 6$, theoretical distribution (10) is also shown (dashed line). The gap amounts to $\varepsilon_{\text{gap}} = \hbar\pi/30\tau_0$ instead of $\varepsilon_{\text{gap}} = \hbar\pi/14\tau_0$ expected from Refs. [6,7,10].

transmission band area. The ratio of the slopes of two lines gives the suppression of the Andreev gap due to the Ehrenfest time fluctuations.

Density of Andreev states.—The area of transmission band equation (8) increases exponentially for small values of λ , while the number of trajectories with small λ is exponentially small (2). These two conflicting effects allow one to find the value of the Lyapunov exponent leading to the largest, for a given time, transmission band. Equating via (10) the total number of expected levels with $\lambda < \lambda_c$ to the area of a single band (8) with $\lambda = \lambda_c$, we find

$$F(\lambda_c) = -\lambda_c + t_D^{-1}. \quad (12)$$

The value of the gap now is

$$\varepsilon_{\text{gap}} = \hbar\pi \frac{\lambda_c}{2 \ln(N\tau_0/t_D)}. \quad (13)$$

The value of λ_c depends on the details of the specific model. It was shown in Ref. [24] that the derivative $F'(\lambda)$ has a maximum at $\lambda = 0$ and that $F'(0) \leq 1$. Since the only maximum of the function F is $F(\lambda_0) = 0$, we obtain

$$\lambda_c < \lambda_0/2. \quad (14)$$

Thus, we found that the value of the Andreev gap is at least twice smaller than predicted previously [6,7,10].

Semiclassical methods may be applied for a description of eigenstates constructed from the trajectories with the Lyapunov exponents

$$\lambda < \lambda_{\text{max}} = t^{-1} \ln(N\tau_0/t_D). \quad (15)$$

For $\lambda > \lambda_{\text{max}}$, the number of levels per transmission band became less than 1. The number of Andreev levels associated with the time t may now be found by integration of Eq. (9) over the interval $\lambda_c < \lambda < \lambda_{\text{max}}$. If $\lambda_{\text{max}} > \lambda_0$, this allows one to recover the known result Eq. (1), which is now valid for $\varepsilon \ln(N\tau_0/t_D) > \hbar\pi\lambda_0$. The smaller energies may exist only due to $\lambda < \lambda_{\text{max}} < \lambda_0$, for which we found a novel form of the density

$$\rho(\varepsilon) \approx \frac{N\tau_0}{\varepsilon t_D F'(\kappa)} \exp \left[\frac{\hbar\pi}{2\varepsilon} F(\kappa) - \frac{\hbar\pi}{2\varepsilon t_D} \right], \quad (16)$$

where $\kappa = (\varepsilon/\hbar\pi) \ln(N\tau_0/t_D)$. Equation (16) is valid for $\lambda_c < \varepsilon \ln(N\tau_0/t_D)/\hbar\pi < \lambda_0$.

Formulas (13) and (16) are the main results of this Letter. They describe not only the stroboscopic model but any Andreev billiard with chaotic dynamics coupled to a superconductor through the N -channel lead. (In this case, τ_0 may be replaced by the averaged time between bouncing of the billiard walls. The precise value of τ_0 is not important since $N \sim \hbar^{-1} \gg t_D/\tau_0$.) For the model (5), the semiclassical density consists of a series of δ -function peaks at $\varepsilon = \varepsilon_{\text{nr}}$ (7), and Eq. (16) describes the smoothed envelope of this distribution. Such peaks in the Andreev spectrum were seen in simulations of Refs. [8,11].

In Eq. (13), we found the averaged value of the superconducting gap. Fluctuations of ε_{gap} are caused by the

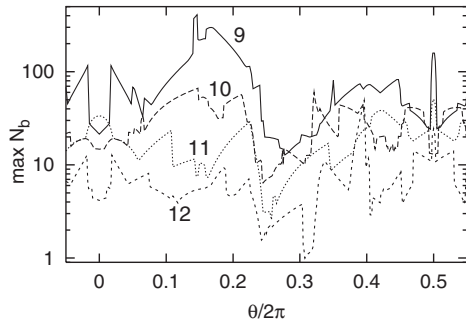


FIG. 3. The area of the largest transmission band (in units of $\hbar = 2\pi \times 10^{-7}$) for 9, 10, 11, and 12 iterations as a function of the position of superconducting contact. Because of the symmetries $\theta \rightarrow -\theta$, $\theta - \pi \rightarrow \pi - \theta$, only half of the $[0, 2\pi]$ range of variation of the argument is shown.

variations in the position of the NS interface. These fluctuations result from the fluctuations of the area of the largest transmission band at a given time, which is shown in Fig. 3. Since the area of (even the largest) transmission band decreases exponentially with time while the number of the bands exponentially increases, we expected that the correlation length for fluctuation of the largest area should also decrease with time. No such increased sensitivity is seen in Fig. 3.

Discussion.—Among the total number of M Andreev levels, M_q levels could not be described semiclassically,

$$M_q \approx M^{1-(1/\lambda_D)}(1 + \ln N/\lambda_D)t_D^{-2/\lambda_D} \gg 1. \quad (17)$$

These levels originate from the phase-space area of the NS interface covered by small transmission bands, $N_b \leq 1$. References [7,10] predict the vanishing of the density of these levels for $\varepsilon < \hbar\pi\lambda_0/2 \ln N$. Although these papers do not provide a rigorous calculation of the Andreev spectrum, we refer to these results as an indication that the levels missing in the adiabatic quantization [6] do not change the low energy density of states (16) found in this Letter.

The density of Andreev levels (16) and the Andreev gap (13), which we found in this Letter, depend on the entire distribution of finite time Lyapunov exponents $P(\lambda, t)$ [or $F(\lambda)$ (2)], not on the most probable Lyapunov exponent λ_0 , as was expected previously [4–10].

The variations of the Lyapunov exponent leading to the fluctuations of Ehrenfest time considered in this Letter are of the order $\sim \lambda_0$. However, the corresponding variations of the extent of the divergency of trajectories become exponentially enhanced $\sim e^{\lambda t}$. So these are indeed the giant fluctuations. The presented results are supported by the classical numerical simulations. Verification of our findings in a real experiment or in quantum-mechanical simulations [26] remains a challenging problem.

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