Abelian and Non-Abelian Hall Liquids and Charge-Density Wave: Quantum Number Fractionalization in One and Two Dimensions

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Previously, we have demonstrated that, on a torus, the Abelian quantum Hall liquid is adiabatically connected to a charge-density wave as the smaller dimension of the torus is varied. In this work, we extend this result to the non-Abelian bosonic Hall state. The outcome of these works is the realization that the paradigms of quantum number fractionalization in one dimension (polyacetylene) and two dimensions (fractional quantum Hall effect) are, in fact, equivalent.

Quantum number fractionalization is a subject of considerable interest in condensed matter physics recently. Two well-known examples where such a phenomenon occurs are (i) the soliton in a one-dimensional (1D) charge-density wave [1] and (ii) the quasiparticles in the fractional quantum Hall effect [2]. Usually, they are regarded as distinct mechanisms of charge fractionalization.

In a recent Letter, Seidel *et al.* have shown that, when placed on a $L_1 \times \infty$ torus, the $\nu = 1/3$ fractional quantum Hall liquid is adiabatically connected to a period-three charge-density wave when L_1 is varied [3]. In a subsequent work, the same conclusion was reached independently [4]. Such continuity is very powerful in identifying discrete quantum numbers, since the latter cannot change during an adiabatic process. For example, the topological threefold degeneracy in the large L_1/l_B limit (l_B is the magnetic length) is adiabatically related to the three charge-density wave patterns at small L_1/l_B. Moreover, the charge $1/3$ quasiparticles at large L_1/l_B are evolved from the Su-Schrieffer domain walls at small L_1/l_B [3].

The above results are based on a 2D → 1D mapping discussed in Refs. [5,6]. Specifically, in the Landau gauge, and after projection to the lowest Landau level, the pseudopotential Hamiltonian [7], for which the Laughlin state is the exact ground state, becomes a center-of-mass position conserving pair hopping model on a 1D lattice [3,5]. The torus parameter L_1/l_B becomes the hopping range. Within this model, it can be shown that for all nonzero L_1/l_B there is an energy gap separating the three degenerate ground states from the excited states. In addition, each ground state has a different center-of-mass position and exhibits a nonzero charge-density wave order as long as L_1/l_B is finite [3,8]. Finally, due to the simultaneous conservation of the center-of-mass position and momentum, all eigenstates are at least threefold degenerate.

In this Letter, we apply the same mapping to study the bosonic non-Abelian quantum Hall state at filling factor $\nu = 1$ [9]. Our motivations are as follows. First, there is a general argument due to Oshikawa [10], which says that, at

filling factor $\nu = p/q$, if a system possesses an excitation gap, then the ground state must be at least q-fold degenerate on a (d-dimensional) torus. From this point of view, one expects that, in the absence of hidden symmetry, the ground state for a gapped $\nu = 1$ system should be nondegenerate. However, based on analyzing the ground state wave function, it was shown that the gapped non-Abelian bosonic Hall state is threefold degenerate on a torus [11,12]. Second, we are curious whether the adiabatic connection in the $\nu = 1/3$ Abelian quantum Hall state also holds true for the non-Abelian case.

In the infinite 2D plane, the bosonic Pfaffian state [9]

$$\Psi = \text{Pf}\left[\frac{1}{z_i - z_j}\right]\prod_{(ij)}(z_i - z_j)\exp\left[-\sum_k |z_k|^2/4\right] \quad (1)$$

is the unique ground state wave function of the following three-particle pseudopotential Hamiltonian at $\nu = 1$:

$$H = \sum_{(ijk)} \delta(z_i - z_j)\delta(z_i - z_k). \quad (2)$$

When the Pfaffian state is generalized to toroidal geometry [11,12], a threefold degeneracy emerges. As emphasized in the beginning, unlike the Abelian quantum Hall state, this degeneracy cannot be attributed to different center-of-mass quantum numbers.

Following our procedure in Ref. [3], we first establish a simple picture for this degeneracy by making the smaller dimension of the torus much smaller than the magnetic length l_B. To this end, we express the Hamiltonian of Eq. (2) in the Landau gauge basis of the lowest Landau level of a $L_1 \times L_2$ torus and obtain $H = \sum_R Q_R^\dagger Q_R$, where

$$Q_R = \sum_{m+n+p=3R \bmod N} f(R - m, R - n, R - p)c_m c_n c_p \quad (3)$$

and $f(a, b, c) = \kappa \sum_{s,t} \exp[-((a + sN)^2 + (b + tN)^2 + (c - (s + t)N)^2)/2]$. In the above, m, n, p runs over the index of the lowest Landau level orbitals. One may identify each orbital with a site on a 1D ring. The total number of

lattice sites N is equal to the number of magnetic flux quanta piercing through the surface of the torus; i.e., $N = L_1 L_2 / 2\pi l_B^2$. The operator c_n annihilates a boson on the nth site. Equation (3) describes center-of-mass conserving triplet hopping. The only parameter κ of this Hamiltonian, $\kappa = 2\pi l_B/L_1$, sets the hopping range.

Symmetry.—The Hamiltonian in Eq. (3) not only commutes with the lattice translation operator T but also commute with the operator

$$U = \exp\left[2\pi i/N \sum_n n c_n^\dagger c_n \right]. \quad (4)$$

We identify U as the center-of-mass position operator of the ring. For filling factor $\nu = p/q$, it is simple to prove that $UT = e^{i2\pi p/q} TU$. Thus, for $\nu = 1$ the two operators T and U commute. As the result, we can label any energy eigenstate by three quantum numbers, i.e., the eigenvalue with respect to H, U, and T, i.e., $|E, u, t\rangle$, where $H|E, u, t\rangle = E|E, u, t\rangle$, $U|E, u, t\rangle = \exp(iu)|E, u, t\rangle$, and $T|E, u, t\rangle = \exp(it)|E, u, t\rangle$.

In addition to the above symmetries, Eq. (3) has the remarkable property that it is "self-dual" under Fourier transformation [3]. That is, when the model is reexpressed in momentum space, which is a lattice of discrete momenta $2\pi n/N$, one obtains a Hamiltonian which is identical to the real space one, except for an inversion of the parameter $\kappa \to \tilde{\kappa} = 2\pi/\kappa N$. Thus, a Hamiltonian at κ in the real space representation is identical to the Hamiltonian at $\tilde{\kappa}$ in momentum space. Consequently, the spectrum of Eq. (3) at κ is identical to that at $\tilde{\kappa} = 2\pi/\kappa N$. Since the quantum numbers u and t interchange upon going from the real to the momentum space and vice versa, it follows that all states at κ that have quantum number $Q = (u, t)$ are degenerate with states at $\tilde{\kappa}$ with quantum number $Q = (t, u)$.

The large κ limit.—Since, for any L_1/l_B, the three Pfaffian ground states [11,12] are zero energy eigenstates of Eq. (2), it follows that for any κ there must be three ground states each annihilated by all the operators Q_R. We will now solve Eq. (3) in the limit $\kappa \gg 1$. By keeping the two largest hopping amplitudes, we obtain

$$H \approx \kappa^2 \sum_n [(c_n^\dagger)^3 (c_n)^3 + \exp(-\kappa^2/3)(c_n^\dagger)^2 c_{n\pm1}^\dagger (c_n)^2 c_{n\pm1}]. \quad (5)$$

Such a Hamiltonian penalizes states that have three particles occupying the same site or three particles occupying two adjacent sites. All other terms in the Hamiltonian are exponentially smaller than the ones retained. The three zero energy states of Eq. (5) are shown in Figs. 1(a)–1(c). Among them, the two states in Figs. 1(a) and 1(b) break the translational symmetry. The third state shown in Fig. 1(c) is translationally *invariant*. These states will be referred to as the (02), (20), and (11) states from now on. Thus, as in the Abelian case [3], we do obtain a simple picture for the ground states in the large κ limit. The

(a) 0202020202020202020202020202020202
(b) 2020202020202020202020202020202020
(c) 1111111111111111111111111111111111
(d) 02020202020202020**1**0202020202020202
(e) 0202020**1**11111111111111111**0**2020202
(f) 11111**0**202020202020202020**1**1111111111
(g) 02020**1**111111**0**20202020**1**111111**0**020202
(h) 02020**1**11111**0**2020202020**1**11111**0**020202
(i) 11111111111**0**2020202020**1**11111111111
(j) 111111111111**0**2020202020**1**1111111111

FIG. 1 (color online). The large κ limits of the (a)–(c) ground state, (d)–(f) and (i),(j) 2-hole excited states, and (g),(h) 4-hole excited state. A vertical segment indicates the position of a charge $\Delta Q = 1/2$ domain wall; see text.

primary question to be answered now is whether the low energy physics at large κ is, in fact, *adiabatically connected* to the limit $\kappa \to 0$, as we argued in the Abelian case.

The degeneracy.—Before we turn to this central issue, we first highlight some crucial differences between the Abelian and the non-Abelian problem. First, while in both cases the correct ground state degeneracy becomes manifest in the large-κ limit, for the Abelian case this degeneracy is dictated by (i) translation invariance, (ii) conservation of the center-of-mass position, and (iii) the noncommutivity between T and U. Thus, it holds true at any value of κ. This argument is no longer valid in the present case, because T and U commute. Furthermore, *all* of the ground states in the Abelian case are related by a simple lattice translation and are, thus, energetically equivalent. This is not true in the present case, since at large κ only two of the ground states are related by translation. Indeed, the (11) state appears as special when compared to the (02) and (20) states, and it seems that a nearest neighbor density-density interaction could lift the threefold degeneracy. While this is true at large κ, we believe that such lifting of degeneracy will become exponentially small as $\kappa \to 0$, and the threefold degeneracy will become robust at small κ. This is because, as we shall show later, the order parameter associated with the (02) and (20) states exponentially vanishes as $\kappa \to 0$.

We now show that, assuming the existence of adiabatic continuity (to be shown below), the threefold ground state degeneracy at any κ can be explained as a consequence of the existence of the twofold degenerate symmetry broken states at large κ *alone*. At large κ, we may form a symmetric and antisymmetric linear combination of the two symmetry breaking ground states, so that they become eigenstates of T with eigenvalues $t = 0$ and $t = \pi$. It is simple to show that both (02) and (20) have eigenvalue $u = 0$ with respect to the center-of-mass operator U. Consequently, the newly formed linear combinations also have the same eigenvalue. Thus, at large κ we have a degenerate pair of ground states with quantum numbers

$Q = (0, 0)$ and $Q = (0, \pi)$. If adiabatic continuity exists, we must have ground states with these quantum numbers at any κ, including $\tilde{\kappa}$. Now let us consider applying the duality transformation to the $Q = (0, \pi)$ ground state at $\tilde{\kappa}$. As we explained earlier, this transforms it to a state $Q = (\pi, 0)$ at κ. Since the ground state energy at κ and $\tilde{\kappa}$ is the same, this transformed state must have the same energy as the $Q = (0, 0)$ and $Q = (0, \pi)$ ground states at κ. Thus, we have generated a third orthogonal state degenerate with the $Q = (0, 0), (0, \pi)$ ground state at the same value of κ. Consequently, the ground state is at least threefold degenerate. In the large κ limit, the $Q = (\pi, 0)$ state is the (11) ground state.

Adiabatic continuity.—Figure 2 displays the charge-density wave order parameter $\mathcal{O} = \sum_n (-1)^n c_n^\dagger c_n / N$ associated with the two symmetry breaking ground states in the $u = 0$ sector of Eq. (3). As explained above, these states are evolved from the (20) and (02) states at large κ. This is manifested by the fact that $|\mathcal{O}|$ becomes 1, reflecting perfect order, in the large κ limit. We note that the value of $\mathcal{O}$ converges remarkably fast with increasing system size (Fig. 2), and it seems hardly necessary to look at larger systems for this purpose. The order parameter seemingly vanishes around $\kappa \approx 0.5$, very similar to the behavior observed in the Abelian case [3]. This could lead one to think that there is a phase transition near this point, below which the order parameter is zero in the $N \to \infty$ limit. However, a careful study of the order parameter at small κ suggests otherwise. In the inset in Fig. 2, we plot the quantity Γ which is related to the order parameter via $\mathcal{O} = \exp(-1/\Gamma^2)$. Although numerical precision limits us to $\kappa \gtrsim 0.29$, we observe that the order parameter continues
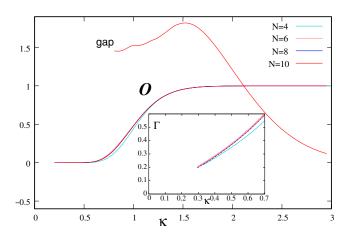
to display a rapidly converging, alternating behavior as a function of system size below $\kappa = 0.5$. The fact that Γ converges to nonzero values as a function of the particle number for all κ values we studied clearly implies that, for these κ values, the limiting value ($N \to \infty$) of the order parameter is nonzero. Moreover, it seems likely that the limiting Γ curve extrapolates into the origin. This implies that as $N \to \infty$ the order parameters associated with the evolved (02) and (20) states stay nonzero except at $\kappa = 0$ (infinite L_1). This is fully consistent with our findings in the Abelian case [3,8]. Thus, the analysis based on the order parameter suggests that there is no phase transition. A new feature that occurs in the non-Abelian case is the existence of a non-symmetry-breaking ground state, i.e., the descendent of the (11) state, at all values of κ. Because of the exponentially small order parameter in Fig. 2, the distinction between the two kinds of ground states becomes academic as κ decreases below 0.5.

Now we demonstrate that the exponentially small order parameter at small κ does not affect the robustness of the energy gap. Figure 2 shows the energy gap for 10 particles as a function of κ. The data is truncated below the self-dual point $\kappa_{sd} = \sqrt{2\pi/N}$ because this part of the data is just the dual image of that above κ_{sd}. In addition, due to the strong size dependence of κ_{sd}, we expect the data below κ_{sd} to display a stronger size dependence and, hence, to be less useful for the purpose of extrapolation. Although, due to the limitation on the particle number, our study of the gap is limited to $\kappa \gtrsim 0.8$, it is apparent that the energy gap does not decline as the order parameter becomes exponentially small. In addition to the above numerical evidence, there is an extra reason to believe that for the bosonic non-Abelian state the gap cannot be caused by symmetry breaking, because there is a totally symmetric ground state at all values of κ. To summarize, both the numerical results for the order parameter and the energy gap are consistent with the notion of adiabatic continuity between small and large κ.

Quasiparticles.—Again, we first understand the quasiparticle and quasihole in the large κ limit. The most naive way to produce a charge $\Delta Q = 1$ state is to remove one boson from one of the ground states, say, the (02), as shown in Fig. 1(d). (Using the nomenclature of the electron, we regard the boson as negatively charged.) As for all the Abelian cases, the positively charged defects cost zero energy. A closer inspection of Fig. 1(d) reveals that such a defect consists of two domain walls [between the ground states in Figs. 1(a) and 1(c)] placed at the closest distance. Since a string of 1's does not cost any additional energy, it is possible to separate the two domain walls, without injecting any extra charge, by converting an even sequence of 0202...02 into a string of 1's as shown in Fig. 1(e). The two domain walls at the end of the string each carry a charge $\Delta Q = 1/2$. Now let us consider removing a boson from the (11) ground state. By turning an odd string of 1's into 0202...20 as shown in Fig. 1(f), two domain walls of

FIG. 2 (color online).   Curves: The charge-density wave order parameter $\mathcal{O}$ of the ground states in the zero center-of-mass sector for $N = 4, 6, 8,$ and 10 particles. These states are adiabatically connected to the (20)- and (02) states in Figs. 1(a) and 1(b). The ground state that is connected to the (11) state in Fig. 1(c) is free of symmetry breaking and, hence, has $\mathcal{O} = 0$. The single curve labeled "gap" shows the energy gap for $N = 10$ versus κ for $\kappa > \kappa_{sd}$. Gap data are scaled down by a factor of 4 for convenience. Inset: The quantity $\Gamma = 1/\sqrt{-\log\mathcal{O}}$.

charge $\Delta Q = 1/2$ are produced. A similar procedure can lead to $\Delta Q = -1/2$, nonzero energy, antidomain walls. Since there is adiabatic continuity, we should be able to tune κ down and let each of these domain walls evolve into the fractional charged quasiholes (delocalized in the L_1 direction) in the non-Abelian Hall liquid. We note that the quasiparticle/quasihole charge of $\Delta Q = \pm 1/2$, which follows naturally from our simple large κ analysis, indeed agrees with the prediction in Ref. [9].

Next we consider four quasiholes. In Figs. 1(g) and 1(h), two distinct four domain wall configurations [on top of the (02) ground state] are presented. These patterns are differentiated by the fact that in Fig. 1(g) [Fig. 1(h)] the middle string of 0,2's are in phase [out of phase] with those at the two ends. To transform Fig. 1(g) to 1(h), one has to move the 1 at the left end of the middle 0,2 string to the right end and shift the entire 0,2 string to the right. For far separated domain walls, this move clearly involves many particles and is very nonlocal. Comparing Figs. 1(g) and 1(h), one might regard the two middle domain walls as having different positions (a shift of one lattice constant). However, this distinction will cease to exist if we let κ decrease so that further range hopping can take place. In that case, each of the domain walls discussed above will smear into a wider and smoother transition region between different ground states. The width of the domain wall increases as κ decreases. Although the tiny difference in the positions of the domain wall becomes immaterial as κ decreases, the fact that in the case of infinitely separated domain walls no finite particle and local moves can transform Figs. 1(g) and 1(h) suggests that they will remain distinct. This results in a twofold degeneracy for four quasihole states at fixed positions. Similarly for six quasiholes, we have three disconnected string of 1's. In the gap between the first and second, and second and third, strings (of 1's), we can again insert either (02) or (20) ground states. Thus, for six quasiholes at fixed positions, we have a degeneracy of 4. Continuing this line of argument, we conclude that, in the presence of $2n$ quasiholes, we have a 2^{n-1}-fold degeneracy. Similar arguments can be carried out for the quasihole injection into the other two ground states. This is consistent with the prediction of the wave function analysis [13].

The above arguments lead us to an interesting observation. There is an intricate difference between the two-domain-wall states in the (02) and (20) background and the (11) background. Let us first focus on the (11) background. Figures 1(i) and 1(j) represent two patterns of domain wall configurations differing by one lattice constant translation. Since we have removed a boson, which changes the filling factor away from 1, the U eigenvalues of these two patterns are different. Hence, they cannot be

mixed by further range hopping when we decrease κ in Eq. (3). Now let us consider the (02) background. It is simple to convince oneself that due to the symmetry breaking and the periodic boundary condition we can only shift the two domain walls by even multiple lattice constants. Consequently, there are twice as many two-domain-wall states in the (11) background as in the (02) and (20) backgrounds. Interestingly, the analysis of Greiter *et al.* shows that for fixed quasihole positions, including all ground states, the two-quasihole state is fourfold rather than three-fold degenerate. We believe that the extra degeneracy is due to the existence of twice as many two-domain wall states that can be added to the (11) ground state.

In conclusion, we have shown that the bosonic non-Abelian Hall liquid is adiabatically connected to simple density patterns as the small dimension of the torus decreases. Together with our earlier result on the Abelian Hall liquid [3,4,8], a powerful conclusion emerges. The paradigms of quantum number fractionalization in one dimension (polyacetylene) and two dimensions (fractionally quantum Hall effect) are, in fact, equivalent.

Note added.—In the process of preparing this manuscript, we became aware of a paper by Bergholtz *et al.* [14] on a closely related subject for the $\nu = 1/2$ fermionic Pfaffian state.

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