Measuring Multipartite Concurrence with a Single Factorizable Observable

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(Received 27 March 2006; published 1 August 2006)

We show that, for any composite system with an arbitrary number of finite-dimensional subsystems, it is possible to directly measure the multipartite concurrence of pure states by detecting only one single factorizable observable, provided that two copies of the composite state are available. This result can be immediately put into practice in trapped-ion and entangled-photon experiments.

DOI: 10.1103/PhysRevLett.97.050501 PACS numbers: 03.67.Mn, 42.50.−p

Introduction.—A number of measures have been proposed to quantify entanglement (see [1] and references therein). Originally defined as an auxiliary quantity for the algebraic evaluation of entanglement of formation of two-qubit systems, concurrence [2] is an entanglement measure in its own right [3]. For the two-qubit case it has a one-to-one correspondence with entanglement of formation [3], and it can be generalized to arbitrary-dimensional bipartite [4,5] and multipartite [6,7] systems. Moreover, for pure states it can be interpreted as the expectation value of a Hermitian operator, and thus it can be measured, if two copies of the state are available [8,9]. As a matter of fact, it was this reinterpretation of concurrence in terms of copies of the state that led to the first direct experimental observation of an entanglement measure [10]. There, a two-qubit entangled state and its copy were encoded in the polarization and transverse momentum degrees of freedom, respectively, of two twin photons generated via parametric down conversion; and concurrence was measured by detecting only a single two-qubit joint probability.

On the other hand, the experimental progress seen in the last few years in the production and coherent manipulation of multiparticle entangled states is tremendous. Three photon W-type entanglement has been observed [11,12]; and three [13,14], four [12,15], and five [16] photon Greenberger-Horn-Zeilinger (GHZ) entangled states are now realizable. A two-atom–one-photon GHZ state has been experimentally demonstrated [17]; three [18], four [19], and up to six [20] ion GHZ states have also been reported; and, very recently, a technique for scalable and deterministic production of W-type entangled states has successfully generated entangled W states of up to eight ions [21]. Nevertheless, there exists a big mismatch between the progress made on the production and manipulation of multiparticle entanglement and its experimental quantification. In the experiments just mentioned, multipartite entanglement was verified either through the use of quantum state tomography [18,21], quantum nonlocality tests [11,13–17], or entanglement witnesses [12,18–21]. Quantum state tomography [22,23] provides a complete description of the state, though is very disadvantageous from the point of view of scalability. Quantum nonlocality tests and entanglement witnesses [24], in contrast, require the measurement of only a few observables; but each of them allows the detection of the entanglement of only a small class of states, so that—in practical terms—some *a priori* knowledge of the state is necessary. And, moreover, they typically provide a qualitative description, but do not define an entanglement measure. Therefore a simple scheme—involving as few measurements as possible—to experimentally measure entanglement of multipartite systems is highly desirable.

In this Letter we show that for a composite system with an arbitrary number of finite-dimensional subsystems it is possible to directly measure the multipartite concurrence of pure states by detecting only one single factorizable observable, provided that two copies of the composite state are available. This allows for a generalization of the single-setting measurement scheme used in [10] to arbitrary-dimensional multipartite systems. In particular, the scheme is directly applicable to trapped-ion and entangled-photon experiments, which we also discuss.

Representation of concurrence using two copies of the state.—It was shown in [9] that the concurrence C_N of an N-partite-system pure state $|\Psi_N\rangle \in \mathcal{H}$, can be expressed as the following expectation value with respect to two copies of $|\Psi_N\rangle$:

$$C_N(\Psi_N) = \sqrt{\langle\Psi_N| \otimes \langle\Psi_N|A|\Psi_N\rangle \otimes |\Psi_N\rangle}. \tag{1}$$

Here A is a Hermitian operator acting on $\mathcal{H} \otimes \mathcal{H}$, i.e., on $\mathcal{H}_1 \otimes \ldots \otimes \mathcal{H}_N \otimes \mathcal{H}_1 \otimes \ldots \otimes \mathcal{H}_N$, where $\mathcal{H}_i$, with $1 \leq i \leq N$, is the Hilbert space associated to the ith subsystem, in terms of which the composite system Hilbert space $\mathcal{H}$ factorizes. The operator A can be written as:

$$A = 4 \sum_{\{s_{j_i}=\pm\}^+} P^1_{s_{1_i}} \otimes \ldots \otimes P^N_{s_{N_i}}, \tag{2}$$

where P^j_+, and P^j_-, ($1 \leq j \leq N$) are the projectors onto the symmetric and antisymmetric subspaces $\mathcal{H}_j \odot \mathcal{H}_j$, and $\mathcal{H}_j \wedge \mathcal{H}_j$, respectively, of the Hilbert space $\mathcal{H}_j \otimes \mathcal{H}_j$ that describes the two copies of the jth subsystem. The antisymmetric subspace is spanned by all states that acquire a phase shift of π upon the exchange of the two involved copies, whereas the symmetric subspace is spanned by those states that acquire no phase shift at all. The summation is restricted to the set $\{s_{j_i} = \pm\}^+$ composed of all possible ways of sorting the symbols "+" and "−" in an N-long string, such that the total amount of − symbols is an even number, and excluding the completely symmetric case with no "−" symbols at all, i.e., $s_{1_i}s_{2_i}\ldots s_{N_i} = ++\ldots+$. The prefactor 4 in Eq. (2) is only a normalization factor so that C_N reduces to the original concurrence [3] for the two-qubit case. When expressed in terms of the reduced density matrices ϱ_i, C_N, as given by Eq. (1), coincides with the multipartite concurrence introduced in [7]:

$$C_N(\Psi_N) = 2^{1-N/2}\sqrt{(2^N - 2) - \sum_i \mathrm{Tr}\varrho_i^2}, \qquad (3)$$

where the index i labels all $(2^N - 2)$ subsets of the N-particle system; and the ϱ_i are the reduced density matrices of all 1 to $N-1$ partite subsystems [9]. C_N vanishes exactly for N-separable states and allows for a meaningful comparison of entanglement between systems with different numbers of subsystems, since the N-partite concurrence $C_N(\Psi_N)$ reduces to the $(N-1)$-partite concurrence $C_N(\Psi_N) = C_{N-1}(\Psi_{N-1})$ for any state $|\Psi_N\rangle = |\Psi_{N-1}\rangle \otimes |\phi\rangle$ that factorizes into an $(N-1)$-partite state and a one-partite remainder. Finally, for $N = 2$, Eq. (3) yields the arbitrary-dimensional bipartite concurrence defined in [4].

C_N in terms of a single factorizable observable.—Any term $P^1_{s_{1_i}} \otimes \ldots \otimes P^N_{s_{N_i}}$ with an even number of antisymmetric factors, projects onto states that are globally symmetric, i.e., that are symmetric with respect to the exchange of the two copies of the entire system, and not only some subsystems. And indeed, the projector $\mathbf{P}_+$ onto the globally symmetric space $\mathcal{H} \odot \mathcal{H}$ is given by the sum over all such terms. In turn, the operator A defined above in Eq. (2) is—up to the prefactor of 4—the projector onto all globally symmetric states with the only exception of those states that are symmetric in every subsystem. Thus, more formally, A reads

$$A = 4(\mathbf{P}_+ - P^1_+ \otimes \ldots \otimes P^N_+). \qquad (4)$$

Now, the twofold copy $|\Psi_N\rangle \otimes |\Psi_N\rangle$ of an arbitrary pure state $|\Psi_N\rangle$—separable or not—is always globally symmetric, i.e., $|\Psi_N\rangle \otimes |\Psi_N\rangle \in \mathcal{H} \odot \mathcal{H}$. On the other hand, any term $P^1_{s_1} \otimes \ldots \otimes P^N_{s_N}$, with an odd number of antisymmetric projectors, projects onto states that are globally antisymmetric, i.e., onto a space that is orthogonal to $\mathcal{H} \odot \mathcal{H}$. Therefore, the expectation value of such a term with respect to a twofold copy $|\Psi_N\rangle \otimes |\Psi_N\rangle$ always vanishes, which is the reason to restrict the sum in Eq. (2) to only terms with an even number of antisymmetric projectors. Thus, one can add to A any contribution of operators that are supported only on $\mathcal{H} \wedge \mathcal{H}$ without changing the value of C_N. In this particular case, it turns out most useful to add the projector $\mathbf{P}_-$ onto the globally antisymmetric space $\mathcal{H} \wedge \mathcal{H}$, weighted with a prefactor 4. Since $\mathbf{P}_-$ and $\mathbf{P}_+$ add up to the identity $\mathbf{1}$, this amounts to replacing A by $\tilde{A}$ in Eq. (1), being

$$\tilde{A} = 4(\mathbf{1} - P^1_+ \otimes \ldots \otimes P^N_+). \qquad (5)$$

Thus, C_N can be expressed in terms of one single *factorizable* observable, which is in contrast to the $2^{N-1} - 1$ terms composing A, required to construct C_N through Eq. (2). Therefore, it can be experimentally determined through the measurement of only one single probability p^N_+ to find each of all N subsystems and their copies in a symmetric state, via

$$C_N(\Psi_N) = 2\sqrt{1 - p^N_+}. \qquad (6)$$

Finally, it is even possible to reduce the number of subsystems on which to measure, as there exists a redundancy in the N-partite measurement. Since the twofold copy $|\Psi_N\rangle \otimes |\Psi_N\rangle$ of a pure state is globally symmetric, it is indeed sufficient to determine the probability of $N-1$ subsystems and copies being in a symmetric state. After the projection of the twofold copies of any $N-1$ subsystems onto their symmetric subspaces, the remaining twofold copy is automatically projected onto its symmetric subspace as well. Thus, the probability of finding all N duplicate subsystems symmetric is equal to its analogous quantity for only $N-1$ subsystems, i.e., $p^N_+ = p^{N-1}_+$.

This redundancy turns out to be very useful to check whether the system is really in a pure state. If one does observe a finite number of events where an odd number of subsystems is in an antisymmetric state, then this indicates that the state in question is not pure, and needs to be described by a density matrix ϱ. The probability of observing an antisymmetric state gives a quantification of the degree of mixing, which can be expressed as $1 - \mathrm{Tr}\varrho^2 = 2\mathrm{Tr}(\mathbf{P}_-\varrho \otimes \varrho)$. Now, analogously to the case of $\mathbf{P}_+$ above, the projector $\mathbf{P}_-$ onto the globally antisymmetric subspace decomposes into all products of P^i_-, and P^i_+ with an odd number of antisymmetric factors. Therefore, adding up the probabilities of observing the corresponding events, allows one to experimentally determine the degree of mixing of ϱ with exactly the same setup used to measure the concurrence.

Application to entangled-photon experiments.—Entangled photons supply us with a system to which the last result is particularly relevant, for the single-setting-measurement scheme used in [10] can be immediately extended to more than two photons. In particular, the experiment that we have in mind is one in which the techniques described in [11–16] to create 3, 4, or even 5 entangled photons are combined with the hyperentanglement techniques described in [25] and used in [10] to create copies of polarization states in the transverse momentum degrees of freedom, so that a multiphoton entangled state is encoded into the photon's polarizations and the copy in the momenta. If such a state is realized with N photons, it is only necessary to perform the two-qubit single-photon Bell-state measurement described in [10] on the polarization-momentum states carried by any $N-1$ of the N photons, to obtain the probability $p_+^{N-1} = p_+^N$ of every polarization qubit and its momentum-qubit copy being in a symmetrical state. Alternatively, other photon spatial degrees of freedom can be used to encode the copy as well, as, for example, the first order Hermite-Gaussian modes, for which unambiguous perfect-efficiency single-photon Bell-state analyzers have also been constructed with linear optical devices [26].

Finally, we emphasize that all measurements are performed locally, as the two qubits are always encoded in the same photon; that the detection in the Bell basis can be done by linear optics Bell-state analyzers; and that discrimination among all four bell states is not required, but rather only between the antisymmetric singlet and the remaining symmetric Bell states. An implementation of our single-setting detection strategy with four or five entangled photons with already existing technology thus seems feasible.

Application to trapped-ion experiments.—Trapped-ions provide another system in which the state-of-the-art of technology allows for an immediate implementation of the scheme developed here. Let us consider an experiment with $2N$ ions trapped in a linear Paul trap with individual laser addressing to each ion. In a first stage of the experiment a GHZ or W state is created in N ions using the techniques described in [18–20] or [21], respectively. In this stage, a collective motional mode is used as the "information bus" among the different ions on which the laser beams shine and is brought to its same initial state in the end. Also, as no laser beam shines on the other N ions, their internal states remain untouched. The result of this first stage is an entangled state encoded into the internal state of the first N ions, and the initial state untouched for the $2N$-ion collective motional mode and internal modes of the second N ions.

Once the entangled state is created in the first N ions, the same procedure is used on the second set of ions to create the copy. After these two stages, the resulting $2N$-ion state is one in which the first N ions share an entangled state and the second N ions share the copy. Then, it is just a matter of choosing any $N-1$ out of the N first ions and measuring each one, with its copy in the second set, in the Bell basis to obtain the probability p_+^{N-1} and thus calculate C_N using Eq. (6). The Bell-state detection, in turn, can be performed by running the sequence of pulses used in [23] backwards, in which all four Bell states were created starting from the product states of the computational basis. In this way, each Bell state can be mapped into a different product state of the computational basis and then finally measured with usual (almost-unit-efficiency) state-selective fluorescence detection [23,27].

Summary.—We showed that for a composite system, with any number of arbitrary-finite-dimensional subsystems, for which a copy of its state is available, it is possible to express the multipartite concurrence of pure states in terms of only one single factorizable observable. This result has immediate utility on trapped-ion and entangled-photon experiments, for which we showed how a direct measurement, with a single experimental setting, of the multipartite concurrence of pure states of photons and ions is feasible with the use of already existing technology.

We would like to thank Alejo Salles and Andreas Buchleitner for helpful discussions, and Stephen Walborn for careful reading of the manuscript. This work was financially supported by the CNPq, the FAPERJ, the CAPES and the Brazilian Millenium Institute for Quantum Information. F. M. gratefully acknowledges financial support by the Alexander von Humboldt Foundation.

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