

Caustic Activation of Rain Showers

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We show quantitatively how the collision rate of droplets of visible moisture in turbulent air increases very abruptly as the intensity of the turbulence passes a threshold, due to the formation of fold caustics in their velocity field. The formation of caustics is an activated process, in which a measure of the intensity of the turbulence, termed the Stokes number St , is analogous to temperature in a chemical reaction: the rate of collision contains a factor $\exp(-C/St)$. Our results are relevant to the long-standing problem of explaining the rapid onset of rainfall from convecting clouds. Our theory does not involve spatial clustering of particles.

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It is common experience that rainfall can commence very abruptly from cumulus clouds (which form when the atmosphere is convecting) but has a much slower onset from stratiform clouds in a stable atmosphere. This can happen even when no part of the cloud is below the freezing point. It is believed that the difference between convecting and stable clouds arises because the convection gives rise to small-scale turbulent motion which facilitates the coalescence of microscopic water droplets (“visible moisture”) into raindrops. This idea has a long history [[1] is a significant contribution containing references to other early papers], but a satisfying theory has been elusive and the topic remains a subject of intensive research, reviewed recently in [2]. Numerical experiments have shown a dramatic increase in the rate of collision of suspended particles when the intensity of turbulence exceeds a certain threshold. This effect was first described in [3]; see also [4]. Here we present a simple quantitative theory of this phenomenon, illustrated in Fig. 1, which shows the collision rate R of an aerosol as a function of a dimensionless parameter termed the Stokes number, St , which contains information about the turbulence intensity, ϵ , and the radius of the water droplets, a (other symbols in the caption are defined later). There is a precipitous increase in the collision rate at a threshold value of St , which was also observed in [3,4]. The current consensus, represented in [2–5], is that the increased rate of collision involves spatial clustering of particles. One exception is [6], which presents a theory having elements (described later) in common with our own, but which is more complex in its formulation and less precise in its conclusions. Spatial clustering plays no role in our theory.

The properties of clouds are very variable and the sizes of visible moisture droplets have a large dispersion, but typically the average of the radius is approximately $10 \mu\text{m}$ and the density is $n \approx 10^8 \text{ m}^{-3}$ [2]. The motion of the droplets is dominated by viscous forces, so that the equation for the position \mathbf{r} of a droplet is well approximated by

$$\ddot{\mathbf{r}} = \gamma(\mathbf{u}(\mathbf{r}, t) - \dot{\mathbf{r}}) \quad (1)$$

until the particles come into contact [1] [here $\mathbf{u}(\mathbf{r}, t)$ is the velocity field of the air and dots denote derivatives with respect to time]. According to Stokes’s formula for the viscous drag on a sphere, $\gamma = 9\rho_g\nu/2\rho_f a^2$ (ν being the kinematic viscosity of air, ρ_f and ρ_g the densities of water and air, respectively): using the values above we estimate $\gamma \approx 500 \text{ s}^{-1}$. The turbulent motion is a multiscale flow with an approximately power-law spectrum of spatial fluctuations of $\mathbf{u}(\mathbf{r}, t)$ [7]. According to the Kolmogorov theory of turbulence, the short wavelength cutoff of the power-law spectrum, η (termed the Kolmogorov length), is a function of ν and of the rate of dissipation per unit mass in the turbulence, ϵ . Similar considerations apply to the shortest characteristic time scale, τ . Dimensional arguments [7] then imply

$$\eta \sim (\nu^3/\epsilon)^{1/4}, \quad \tau \sim (\nu/\epsilon)^{1/2}. \quad (2)$$

The typical velocity difference for points separated by the Kolmogorov length η will be denoted by u_0 . The dimensional arguments of the Kolmogorov theory imply that

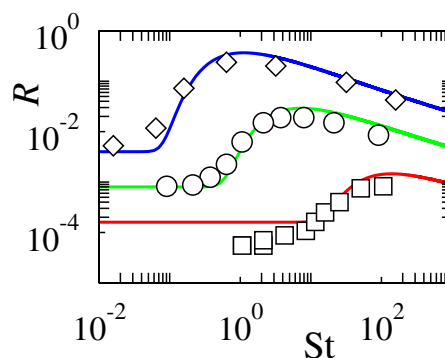


FIG. 1 (color online). Particle collision rate as a function of Stokes number, for three different values of the Kubo number [defined in Eq. (3)]. In these simulations [described in full below Eq. (7)] we used $n = 10^3$, $a = 2.5 \times 10^{-4}$, $\eta = 0.1/\pi$, $\tau = 0.1$, and varied u_0 and γ : $Ku = 1$ (\diamond), $Ku = 0.2$ (\circ), and $Ku = 0.04$ (\square). The theoretical curves are (7), with actions S obtained from Fig. 3. The constant C_a in $R_a = C_a n a^d u_0 / \eta$ was fitted (see text).

$u_0 \sim \eta/\tau$ for turbulent flow. The dissipation rate ϵ is highly variable between different clouds. Values of $\epsilon = 0.1 \text{ m}^3 \text{ s}^{-2}$ are typical [2]; $\nu \approx 10^{-5} \text{ m}^2 \text{ s}^{-1}$ for air in the lower atmosphere, giving $\eta \approx 3 \times 10^{-4} \text{ m}$, $\tau \approx 10^{-2} \text{ s}$.

We can form four independent dimensionless parameters describing the microscopic motion of the water droplets: these are

$$\text{St} = 1/\gamma\tau, \quad \text{Ku} = u_0\tau/\eta, \quad n\eta^d, \quad \eta/a \quad (3)$$

(d is the dimensionality of space). The values quoted above give $\text{St} \approx 0.2$. The Kubo number Ku cannot be large [8] and the comments above indicate that it is of order unity for fully developed turbulence, but small values of Ku occur in stirred fluids, and this parameter will play a role later. From the data above we see that $n\eta^d \ll 1$ and $\eta/a \gg 1$ for water droplets in clouds.

Saffman and Turner [1] only considered the case of $\text{St} \ll 1$ in detail, because they believed that $\text{St} \gg 1$ would only be realized in the most unstable cumulonimbus clouds. Later it was understood that fully developed turbulent motion with high Reynolds number can exhibit pronounced intermittency [7]. One consequence is that the rate of dissipation can fluctuate wildly relative to the mean value, and measurements in clouds have indicated that ϵ can have an approximately log-normal distribution [2]. The notion that intermittency can promote the formation of rain showers by creating localized regions of very high turbulent intensity is considered in [2,6].

Aerosol particles in a turbulent flow may exhibit a degree of clustering: mechanisms for this process were proposed in [5,6,9] and Ref. [8] describes recent advances. Most of the recent literature on the initiation of rainfall by turbulence assumes that the mechanism involves clustering [2–5]; however, this theory is unsatisfactory for two reasons. First, the widely accepted view is that clustering is due to particles being centrifuged away from regions of high vorticity. It is argued that this effect is strongest when St is close to unity: it does not occur when the particle motion is highly damped ($\text{St} \ll 1$), or when the vortices are too short lived ($\text{St} \gg 1$). Figure 1 shows, however, that the collision rate rises abruptly at a threshold value of St (which depends upon Ku) and remains high for $\text{St} \gg 1$: this is equally true for simulations of single-scale flows, such as Fig. 1, and multiscale flows, such as those in [3]. The other difficulty is that the clustering can only occur for particle separations smaller than η , but in a cloud the density of particles is so low that there is unlikely to be more than one particle in a cube of length η .

In our theory, the dramatic increase in the collision rate is a consequence of the formation of “caustics,” illustrated schematically in Fig. 2. Here the velocity of the water droplets as a function of position is initially single valued [curve (a)] but droplets with a large velocity overtake slower-moving particles, so that at a later time the droplet velocity is multivalued [curve (b)]. The region where the

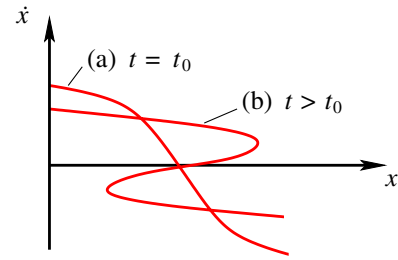


FIG. 2 (color online). Schematic showing the formation of a caustic. (a) Initial configuration, showing droplet velocity as a function of position along an arbitrary line. (b) This manifold has developed fold caustics. The velocity field is multivalued between the folds.

velocity is a multivalued function is bounded by two fold caustics. The caustics have two effects which could enhance the collision rate of water droplets. First, at the caustic lines themselves there can be a divergence in the density of particles. This effect is discussed in [6,10]: it is analogous to the divergence of light intensity on optical caustics [11], but we argue that it is not relevant to collisions between microscopic water droplets in clouds, where $n\eta^3 \ll 1$. The other effect of the caustics is that when the velocity field is multivalued, droplets at the same position are moving with differing velocity, and their relative motion produces collisions. This has no analogue in optical caustics. The importance of this effect was previously emphasized in [6].

Our theory must consider the rate of collision between droplets both with and without caustics, and the rate at which caustics are formed. Saffman and Turner [1] discussed the collision (due to shearing motion) of droplets which are advected with the air: this approximation is valid in the limit as $\text{St} \rightarrow 0$; we term this advective collision rate R_a . At large Stokes numbers a theory due to Abrahamson [12] treats the water droplets as a gas of particles with random and uncorrelated motion and calculates the collision rate R_g by gas-kinetic theory. This theory is widely regarded as being applicable only at very large Stokes numbers, and indeed the two theories give results which differ by a factor of order η/a at $\text{St} \approx 1$. This might be taken as evidence that at least one of these theories does not work when the Stokes number is of order unity. However, we shall argue that the gas-kinetic model is applicable as soon as caustics have formed and that the collision rate is well approximated by $R = R_a + f(\text{St}, \text{Ku})R_g$, where $f(\text{St}, \text{Ku})$ is the fraction of the coordinate space for which the velocity field has become multivalued due to the formation of caustics.

First we consider the rate of production of caustics in more depth. In earlier works [10,13], two of the present authors described the formation of caustics in the limit where $\text{Ku} \ll 1$. In the one-dimensional case, we considered the linearization of Eq. (1), describing the small separation in space δx and velocity δv for two nearby

droplets. The equation of motion for $X = \delta v / \delta x$ is [14]

$$\frac{dX}{dt} = -\gamma X - X^2 + \frac{\partial u}{\partial x}(x(t), t) \quad (4)$$

which is a stochastic differential equation, in which the velocity gradient acts as a random forcing term. The droplets encounter a caustic whenever δx passes through zero, implying that X goes to infinity in one direction, then jumps instantaneously to the reflected point at infinity. When fluctuations drive X to a sufficiently negative value, it will almost certainly escape to $-\infty$ in a finite time. In [13], Eq. (4) was considered in the limit $\text{St} \rightarrow \infty$, $\text{Ku} \rightarrow 0$, where the dynamics of X can be approximated by a Langevin process. The associated Fokker-Planck equation was solved exactly, giving the rate J at which any one droplet passes through caustics. Similar results were obtained in two [10] and three [8] spatial dimensions. They are expressed in terms of a dimensionless parameter I defined in terms of the strain-rate correlation function

$$I = \frac{1}{2\gamma} \int_{-\infty}^{\infty} dt \left\langle \frac{\partial u_1}{\partial x_1}(\mathbf{r}(t), t) \frac{\partial u_1}{\partial x_1}(\mathbf{r}(0), 0) \right\rangle \quad (5)$$

(we use angular brackets to indicate averages). In the limit as $I \rightarrow 0$, we find that the rate of caustic formation is determined by the rate of escape due to diffusion of the coordinate X from an attractive fixed point at $X = 0$. The escape rate is asymptotic to

$$J = J_0 \exp(-S/I) \quad (6)$$

(for some constant J_0), where in S can be obtained as the action of a trajectory of a Hamiltonian function [15,16]. Note that we can write $I \sim \text{Ku}^2 \text{St}$, so that (6) is consistent with the alternative form quoted in the abstract. In one dimension we extract $S = 1/6$ from our exact expression for J , obtained in [13]; further results for one dimension are given in [17]. In two and three dimensions, one obtains $S \approx 0.14$ [10] and $S \approx 0.12$ [8] in the limit as $\text{Ku} \rightarrow 0$. Our numerical results, summarized in Fig. 3, show that the activated escape model for the formation of caustics, described by (6), is valid even when Ku approaches unity. We find that the action S depends upon Ku . For our model (described below) we found $S = 0.70$ for $\text{Ku} = 1$, $S = 0.18$ for $\text{Ku} = 0.2$, and $S = 0.15$ for $\text{Ku} = 0.04$, consistent with the limiting value $S = 0.14$ for $\text{Ku} \rightarrow 0$ quoted above.

The dependence of J upon I (or equivalently, on St) is analogous to the dependence of the rate of a chemical reaction on temperature T , which is well approximated by expressions containing an Arrhenius factor, $\exp(-E/kT)$ (E is the activation energy, k is the Boltzmann constant). In this sense we may regard the formation of caustics as an activated process, in which the dimensionless intensity of the turbulence (I or St) plays a role analogous to temperature.

We argue that the function $f(\text{St}, \text{Ku})$ describing the prevalence of caustics is closely related to the rate of

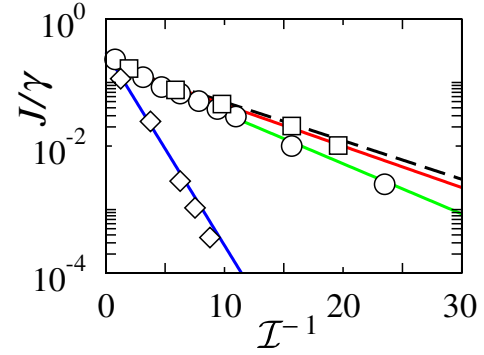


FIG. 3 (color online). Rate of caustic formation as a function of $I \propto \text{Ku}^2 \text{St}$, showing evidence for “activated” behavior: $\text{Ku} = 1$ (\diamond), with action $S = 0.70$, $\text{Ku} = 0.2$, $S = 0.18$ (\circ), and $\text{Ku} = 0.04$, $S = 0.15$ (\square). Also shown is the limiting behavior as $\text{Ku} \rightarrow 0$, with action $S = 0.14$ (dashed line).

caustic formation, so that the collision rate is well approximated by

$$R = R_a + \exp(-S/I) R_g. \quad (7)$$

Formulae for the advective and gas-kinetic collision rates, R_a and R_g respectively, are given below. It is Eq. (7) which is the theoretical curve illustrated in Fig. 1, using values of S obtained from Fig. 3. We fitted the prefactor of the expression for R_a , which is the dominant term when $\text{St} \ll 1$, but there are no other fitted parameters. We remark that combining Eqs. (2), (4) of [6] gives an expression which can be written in the form $R = R_a + PR_c$, analogous to our (7), in which R_c is similar to the precise asymptote, R_g . The expression for the factor P in [6] has a different dependence upon St .

Our numerical simulations use a two-dimensional model in which the divergenceless velocity $\mathbf{u}(\mathbf{x}, t)$ is obtained from a scalar random field $\psi(\mathbf{x}, t)$ by writing $\mathbf{u} = (\partial\psi/\partial y, -\partial\psi/\partial x)$. The field $\psi(\mathbf{x}, t)$ is a Gaussian random function with spatially and temporally stationary, isotropic statistics. The mean value is zero and the correlation function is $\langle \psi(\mathbf{x}, t) \psi(\mathbf{x}', t') \rangle = u_0^2 \eta^2 \exp[-(t-t')^2/2\tau^2] \times \exp(-|\mathbf{x}' - \mathbf{x}|^2/2\eta^2)$, which gives $I = \sqrt{\pi}/2 \text{Ku}^2 \text{St}$ for this model. The droplets are initially randomly positioned, and they are regarded as having collided when their separation falls below $2a$ [that is, we assume that the collision efficiency is unity [1]]. Figure 1 shows the rate of collision R for a single particle. We did not include gravitational settling or other effects which also occur without turbulence. The results for our single-scale flow are very similar to the simulations using Navier-Stokes flows, reported in [3,4], implying that the multiscale aspect of those simulations is not essential to their understanding.

In order to complete the discussion of our formula (7) for the collision rate we describe the asymptotic formulas for the advective and gas-kinetic collision rates. In the advective limit the suspended particles are brought into contact

by the effect of shearing motion in the flow. The typical shear rate is u_0/η . The relative velocity for particles that will be brought into contact is $\Delta v \sim au_0/\eta$, and the volume swept in time Δt is $\Delta V \sim a^d u_0 \Delta t/\eta$. Collisions typically occur when $n\Delta V \sim 1$, so that the expected rate of collision is $R_a = C_a n a^d u_0/\eta$ (where C_a is a constant which we fitted numerically). Saffman and Turner [1] gave an upper bound on the collision rate which can be calculated analytically if correlation functions of the flow field are known. Their approach gives a precise asymptote for the steady hyperbolic flow which they discussed, but in general it is an upper bound because their calculation does not account for multiple collisions in generic flows (which are not steady and which may not be locally hyperbolic).

When $St \gg 1$, the inertia of the aerosol particles means that their motion becomes uncorrelated with the velocity of fluid, and a gas-kinetic model of the type proposed by Abrahamson [12] is appropriate. The maximal Lyapunov exponent λ for motion of the particles is positive [15] and if the rate of collision satisfies $R \ll \lambda$, then the velocities of nearby particles are completely randomized by the time they collide (our estimates below verify this assumption). The aerosol particles become a gas of droplets with velocities which are uncorrelated with each other and with that of the turbulent air. The velocity distribution is Maxwellian, even if \mathbf{u} is not Gaussian distributed, because the equation of motion for the droplet velocity is analogous to an Ornstein-Uhlenbeck process [18]. The rate of collision is exactly the same as for a hard-sphere gas with the same particle radius and rms velocity. The number of collisions per unit time of a given particle in a dilute gas is $R_g \sim n a^{d-1} \bar{v}_r$ ($R_g = 4n a \bar{v}_r$ in two dimensions), where \bar{v}_r is the mean relative speed of the droplets. For a Maxwellian velocity distribution, $\bar{v}_r = \sqrt{\pi/4} \sqrt{\langle \mathbf{v}_r^2 \rangle}$ in two dimensions and we have $R_g = 2\sqrt{\pi} n a \sqrt{\langle \mathbf{v}_r^2 \rangle}$. It remains to calculate the mean-squared velocity of the water droplets, $\langle \mathbf{v}^2 \rangle$, from which we obtain $\langle \mathbf{v}_r^2 \rangle = 2\langle \mathbf{v}^2 \rangle$. The solution of the equation of motion (1) for the velocity of a droplet is

$$\mathbf{v}(t) = \gamma \int_{-\infty}^t dt' \exp[-\gamma(t-t')] \mathbf{u}(\mathbf{r}(t'), t'). \quad (8)$$

The variance of the velocity is obtained by squaring this expression and averaging: when $St \gg 1$ this is asymptotic to

$$\langle \mathbf{v}^2 \rangle = \frac{\gamma}{2} \int_{-\infty}^{\infty} dt \langle \mathbf{u}(\mathbf{0}, t) \cdot \mathbf{u}(\mathbf{0}, 0) \rangle. \quad (9)$$

This gives a collision rate $R_g \sim n a^{d-1} u_0 \sqrt{\gamma \tau}$ (for our

two-dimensional model, $R_g = 2^{7/4} \pi^{3/4} n a u_0 \sqrt{\gamma \tau}$). Thus R_g/R_a is of order η/a when $St \approx 1$ and $Ku \approx 1$.

In summary, we have argued that the collision rate is well approximated by the gas-kinetic model as soon as the turbulence intensity I exceeds the action S . Finally, we must consider whether this gives a sufficiently rapid rate of collision. In three dimensions, the gas-kinetic model gives a collision rate $R_g = 16\sqrt{\pi/3} n a^2 \sqrt{\langle \mathbf{v}^2 \rangle}$. From (9), we estimate $\sqrt{\langle \mathbf{v}^2 \rangle} \sim \sqrt{St} \eta/\tau$. Using the above data, this estimate gives a collision rate of $R_g \approx 5 \times 10^{-3} \text{ s}^{-1}$ when $St \approx 1$. We conclude that rainfall can be initiated in a time scale of a few minutes, provided that a sufficiently large part of the cloud has a turbulence intensity, I which exceeds the action for forming caustics, S .

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