

## Strong-Coupling Constant with Flavor Thresholds at Five Loops in the Modified Minimal-Subtraction Scheme

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We present in analytic form the matching conditions for the strong-coupling constant  $\alpha_s^{(n_f)}(\mu)$  at the flavor thresholds to four loops in the modified minimal-subtraction scheme. Taking into account the present knowledge on the coefficient  $\beta_4$  of the Callan-Symanzik beta function of quantum chromodynamics, we thus derive a five-loop formula for  $\alpha_s^{(n_f)}(\mu)$  together with appropriate relationships between the asymptotic scale parameters  $\Lambda^{(n_f)}$  for different numbers of flavors  $n_f$ .

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The strong-coupling constant  $\alpha_s^{(n_f)}(\mu) = g_s^2/(4\pi)$ , where  $g_s$  is the gauge coupling of quantum chromodynamics (QCD), is a fundamental parameter of the standard model of elementary particle physics; its value  $\alpha_s^{(5)}(M_Z)$  is listed among the constants of nature in the Review of Particle Physics [1]. Here,  $\mu$  is the renormalization scale, and  $n_f$  is the number of active quark flavors  $q$ , with mass  $m_q \ll \mu$ . The  $\mu$  dependence of  $\alpha_s^{(n_f)}(\mu)$  is controlled by the Callan-Symanzik beta function of QCD,

$$\begin{aligned} \mu^2 \frac{d}{d\mu^2} \frac{\alpha_s^{(n_f)}(\mu)}{\pi} &= \beta^{(n_f)} \left( \frac{\alpha_s^{(n_f)}(\mu)}{\pi} \right) \\ &= - \sum_{N=0}^{\infty} \beta_N^{(n_f)} \left( \frac{\alpha_s^{(n_f)}(\mu)}{\pi} \right)^{N+2}. \end{aligned} \quad (1)$$

The calculation of the one-loop coefficient  $\beta_0^{(n_f)}$  about 33 years ago [2] has led to the discovery of asymptotic freedom and to the establishment of QCD as the theory of strong interactions, an achievement that was awarded by the 2004 Nobel Prize in Physics. In the class of schemes where the beta function is mass independent, which includes the minimal-subtraction (MS) schemes of dimensional regularization [3],  $\beta_0^{(n_f)}$  and  $\beta_1^{(n_f)}$  [4] are universal. The results for  $\beta_2^{(n_f)}$  [5] and  $\beta_3^{(n_f)}$  [6] are available in the modified MS ( $\overline{\text{MS}}$ ) scheme [7]. As for  $\beta_4^{(n_f)}$ , the term proportional to  $n_f^4$ ,

$$\beta_4^{(n_f)} = \left[ \frac{1205}{2\,985\,984} - \frac{19}{10\,368} \zeta(3) \right] n_f^4 + O(n_f^3), \quad (2)$$

where  $\zeta$  is Riemann's zeta function, was found in the large- $n_f$  expansion [8], while the residual terms of  $O(n_f^3)$  and below are presently unknown. However, the latter were estimated by an educated guess, through weighted asymptotic Padé approximant predictions (WAPAPs), which are improved by including asymptotic corrections with respect to the usual Padé approximants and performing a weighted average over negative values of  $n_f$  [9]. In the case of  $\beta_3^{(n_f)}$ , leaving aside the quartic Casimir terms, which appear there

for the first time, the WAPAPs approximate the exact coefficients of  $n_f^n$  with  $n = 0, 1, 2$  amazingly well, at the 1% level. One may thus expect that the WAPAPs for  $\beta_4^{(n_f)}$  work similarly well, except for the quartic Casimir terms, which cannot be predicted quite as reliably. For the reader's convenience,  $\beta_N^{(n_f)}$  ( $N = 0, \dots, 4$ ) are listed for the  $n_f$  values of practical interest in Table I.

In MS-like renormalization schemes, the Appelquist-Carazzone decoupling theorem [10] does not in general apply to quantities that do not represent physical observables, such as beta functions or coupling constants; i.e., quarks with mass  $m_q \gg \mu$  do not automatically decouple. The standard procedure to circumvent this problem is to render decoupling explicit by using the language of effective field theory. As an idealized situation, consider QCD with  $n_l = n_f - 1$  massless quark flavors and one heavy flavor  $h$ , with mass  $m_h \gg \mu$ . Then, one constructs an effective  $n_l$ -flavor theory by requiring consistency with the full  $n_f$ -flavor theory at the heavy-quark threshold  $\bar{\mu} = O(m_h)$ . This leads to a nontrivial matching condition between the couplings of the two theories. Although,  $\alpha_s^{(n_l)}(m_h) = \alpha_s^{(n_f)}(m_h)$  at leading and next-to-leading orders, this relationship does not generally hold at higher orders in the  $\overline{\text{MS}}$  scheme; i.e.,  $\alpha_s^{(n_f)}(\mu)$  starts to exhibit finite discontinuities at the flavor thresholds. If the  $\mu$  evolution of  $\alpha_s^{(n_f)}(\mu)$  is to be performed at  $N + 1$  loops, i.e., with the highest coefficient in Eq. (1) being  $\beta_N^{(n_f)}$ , then consistency requires that the matching conditions be im-

TABLE I.  $\overline{\text{MS}}$  values of  $\beta_N^{(n_f)}$  for variable  $n_f$ .  $\beta_4^{(n_f)}$  is estimated by WAPAPs with quartic Casimir terms omitted [9].

$n_f$	$\beta_0^{(n_f)}$	$\beta_1^{(n_f)}$	$\beta_2^{(n_f)}$	$\beta_3^{(n_f)}$	$\beta_4^{(n_f)}$
3	$\frac{9}{4}$	4	$\frac{3863}{384}$	$\frac{445}{32} \zeta(3) + \frac{140\,599}{4608}$	162
4	$\frac{25}{12}$	$\frac{77}{24}$	$\frac{21\,943}{3456}$	$\frac{78\,535}{5184} \zeta(3) + \frac{4\,918\,247}{373\,248}$	119
5	$\frac{23}{12}$	$\frac{29}{12}$	$\frac{9769}{3456}$	$\frac{11\,027}{648} \zeta(3) - \frac{598\,391}{373\,248}$	107
6	$\frac{7}{4}$	$\frac{13}{8}$	$-\frac{65}{128}$	$\frac{11\,237}{576} \zeta(3) - \frac{63\,559}{4608}$	124

plemented in terms of  $N$ -loop formulas. Then, the residual  $\mu$  dependence of physical observables will be of order  $N + 2$ . The QCD matching conditions at the flavor thresholds to two [11] and three [12] loops are known in analytical form; they are routinely used in the literature and even copied to the Review of Particle Physics [1]. Recently, the four-loop result was found, in semianalytical form [13]. In fact, the most intricate four-loop tadpole master integrals involving one nonvanishing mass among the basic set that enters any such calculation could so far only be computed numerically, with limited precision [13,14]. It is the purpose of this Letter to overcome this bottleneck by presenting the four-loop matching condition for  $\alpha_s^{(n_f)}(\mu)$  entirely in terms

of standard transcendental numbers. This requires the analytic evaluation of the massive four-loop tadpole diagram that is called  $X_0$  or  $T_{91}$  in the recent literature [13]. Together with the results of Ref. [15], we thus enhance the knowledge of the basic set of massive four-loop tadpole master integrals in analytic form.

Prior to explaining the core of this analysis and presenting our analytic result for the four-loop matching condition for  $\alpha_s^{(n_f)}(\mu)$ , we derive the five-loop formula for this coupling for fixed value of  $n_f$ . In order to simplify the notation, we introduce the couplant  $a^{(n_f)}(\mu) = \alpha_s^{(n_f)}(\mu)/\pi$  and omit the labels  $\mu$  and  $n_f$  wherever confusion is impossible. Integrating Eq. (1) leads to

$$\ln \frac{\mu^2}{\Lambda^2} = \int \frac{da}{\beta(a)} = \frac{1}{\beta_0} \left[ \frac{1}{a} + b_1 \ln a + a(-b_1^2 + b_2) + a^2 \left( \frac{b_1^3}{2} - b_1 b_2 + \frac{b_3}{2} \right) + a^3 \left( -\frac{b_1^4}{3} + b_1^2 b_2 - \frac{b_2^2}{3} - \frac{2}{3} b_1 b_3 + \frac{b_4}{3} \right) + O(a^4) \right] + C, \quad (3)$$

where  $b_N = \beta_N/\beta_0$  ( $N = 1, \dots, 4$ ),  $\Lambda$  is the so-called asymptotic scale parameter, and  $C$  is an arbitrary constant. The second equality in Eq. (3) is obtained by expanding the integrand. The conventional  $\overline{\text{MS}}$  definition of  $\Lambda$ , which we adopt, corresponds to choosing  $C = (b_1/\beta_0) \ln \beta_0$  [7,16]. Iteratively solving Eq. (3) yields, with  $L = \ln(\mu^2/\Lambda^2)$ ,

$$a = \frac{1}{\beta_0 L} - \frac{b_1 \ln L}{(\beta_0 L)^2} + \frac{1}{(\beta_0 L)^3} [b_1^2 (\ln^2 L - \ln L - 1) + b_2] + \frac{1}{(\beta_0 L)^4} \left[ b_1^3 \left( -\ln^3 L + \frac{5}{2} \ln^2 L + 2 \ln L - \frac{1}{2} \right) - 3b_1 b_2 \ln L + \frac{b_3}{2} \right] + \frac{1}{(\beta_0 L)^5} \left[ b_1^4 \left( \ln^4 L - \frac{13}{3} \ln^3 L - \frac{3}{2} \ln^2 L + 4 \ln L + \frac{7}{6} \right) + 3b_1^2 b_2 (2 \ln^2 L - \ln L - 1) - b_1 b_3 \left( 2 \ln L + \frac{1}{6} \right) + \frac{5}{3} b_2^2 + \frac{b_4}{3} \right] + O\left(\frac{1}{L^6}\right). \quad (4)$$

The particular choice of  $C$  [7,16] in Eq. (3) is predicated on the grounds that it suppresses the appearance of a term proportional to  $(\text{const}/L^2)$  in Eq. (4).

We now turn to the analytic evaluation of the four-loop matching condition for  $\alpha_s^{(n_f)}(\mu)$  at the flavor thresholds. The underlying formalism was comprehensively explained in Refs. [11,12], and most of the technical issues related to its application at four loops were already discussed in Ref. [13]. For lack of space, we thus concentrate here on the missing link of this analysis beyond the scope of Ref. [13], namely, the analytic evaluation of the massive four-loop tadpole diagram  $X_0$ , which is depicted in Fig. 1(a). This task may be simplified by noticing that  $X_0$  does not represent a master integral, but may be reduced to simpler integrals with less lines, all of which are analytically known [14], some for a short time only [15], except for the one ( $J_0$ ) shown in Fig. 1(b). The integral  $J_0$  is finite, and the coefficients of its expansion in  $\epsilon$ , where  $D = 4 - 2\epsilon$  is the dimensionality of space-time, have only one level of transcendentality [17]; i.e., they contain polylogarithms  $\text{Li}_k$  and zeta functions  $\zeta(k)$  with the same value of  $k$ . These properties reduce the number of terms and thus simplify the calculation. In order to evaluate  $J_0$ , we temporarily introduce an artificial mass splitting among the four massive lines in Fig. 1(b), by assigning the mass  $m$  to any two of them and the mass  $M$  to the other two. We then perform an expansion in the ratio  $x = m^2/M^2$  using the large-mass expansion technique and recover the complete series in  $x$  as explained in Ref. [17]. Through  $O(\epsilon^2)$ , we have

$$(1 - \epsilon)(m^2 M^2)^{2\epsilon} m^2 J_0 = \sum_{n=1}^{\infty} x^n \left\{ -\frac{8}{(2n-1)^3} + 16\epsilon \left[ -\frac{\zeta(3) - \tilde{S}_3}{2n-1} + \frac{2\tilde{S}_2}{(2n-1)^2} + \frac{4\tilde{S}_1}{(2n-1)^3} \right] + 16\epsilon^2 \left[ \frac{9\zeta(4) + 8\zeta(3)\tilde{S}_1 - 4\tilde{S}_2^2 - 8\tilde{S}_1\tilde{S}_3 - 3\tilde{S}_4}{2n-1} + \frac{4\zeta(3) - 16\tilde{S}_1\tilde{S}_2 - 4\tilde{S}_3}{(2n-1)^2} - \frac{\zeta(2) + 16\tilde{S}_1^2 + 4\tilde{S}_2}{(2n-1)^3} - \frac{2}{(2n-1)^5} \right] + O(\epsilon^3) \right\}, \quad (5)$$

where we introduced the shorthand notation  $\tilde{S}_a = 2^{a-2} S_a(2n-1) - S_a(n-1)$ , with  $S_{\pm a}(n) = \sum_{j=1}^n (\pm 1)^j j^{-a}$  being harmonic sums, and omitted irrelevant terms involving  $\ln x$ . We then put  $x = 1$  and exploit the identities

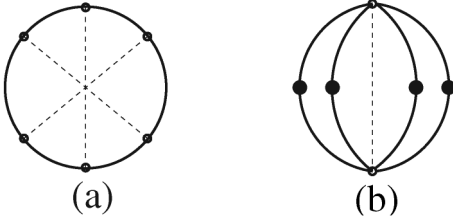


FIG. 1. Four-loop tadpole diagrams (a)  $X_0$  and (b)  $J_0$ . Dashed and solid lines represent massless and massive propagators, respectively; a dot on a line duplicates that propagator.

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{\tilde{S}_a}{(2n-1)^c} &= -\frac{2^a}{8} \sum_{l=1}^{\infty} \frac{1-(-1)^l}{l^c} [S_a(l) + 2S_{-a}(l)], \\ \sum_{n=1}^{\infty} \frac{\tilde{S}_a \tilde{S}_b}{(2n-1)^c} &= \frac{2^{a+b}}{32} \sum_{l=1}^{\infty} \frac{1-(-1)^l}{l^c} [S_a(l) + 2S_{-a}(l)] \\ &\quad \times [S_b(l) + 2S_{-b}(l)], \end{aligned} \quad (6)$$

at transcendentality levels  $k = a + c$  and  $k = a + b + c$ ,

$$\begin{aligned} \frac{a'}{a} &= 1 - a \frac{\ell}{6} + a^2 \left( \frac{\ell^2}{36} - \frac{11}{24} \ell + c_2 \right) + a^3 \left[ -\frac{\ell^3}{216} + \ell^2 \left( \frac{53}{576} - \frac{n_l}{36} \right) + \ell \left( -\frac{955}{576} + \frac{67}{576} n_l \right) + c_3 \right] \\ &\quad + a^4 \left\{ \frac{\ell^4}{1296} + \ell^3 \left( -\frac{1883}{10368} - \frac{127}{5184} n_l + \frac{n_l^2}{324} \right) + \ell^2 \left( \frac{2177}{3456} - \frac{1483}{10368} n_l - \frac{77}{20736} n_l^2 \right) \right. \\ &\quad \left. + \ell \left[ \frac{7391699}{746496} - \frac{2529743}{165888} \zeta(3) + n_l \left( -\frac{110341}{373248} + \frac{110779}{82944} \zeta(3) \right) + \frac{6865}{186624} n_l^2 \right] + c_4 \right\} + O(a^5), \end{aligned} \quad (10)$$

where  $\ell = \ln[\bar{\mu}^2/m_h^2(\bar{\mu})]$  and

$$\begin{aligned} c_2 &= \frac{11}{72}, \quad c_3 = \frac{564731}{124416} - \frac{82043}{27648} \zeta(3) - \frac{2633}{31104} n_b, \\ c_4 &= \frac{291716893}{6123600} - \frac{2362581983}{87091200} \zeta(3) - \frac{76940219}{2177280} \zeta(4) + \frac{9318467}{362880} \zeta(4) \ln 2 - \frac{12057583}{483840} \zeta(5) + \frac{3031309}{435456} a_4 + \frac{340853}{90720} a_5 \\ &\quad + n_l \left( -\frac{4770941}{2239488} + \frac{3645913}{995328} \zeta(3) - \frac{541549}{165888} \zeta(4) + \frac{115}{576} \zeta(5) + \frac{685}{41472} a_4 \right) + n_l^2 \left( -\frac{271883}{4478976} + \frac{167}{5184} \zeta(3) \right). \end{aligned} \quad (11)$$

The counterpart of Eq. (10) in the on-shell scheme of mass renormalization is obtained by substituting the three-loop relationship between  $m_h(\mu)$  and the pole mass  $M_h$  [18].

Going to higher orders, one expects, on general grounds, that the relationship between  $\alpha_s^{(n_l)}(\mu')$  and  $\alpha_s^{(n_l)}(\mu)$ , where  $\mu' \ll \bar{\mu} \ll \mu$ , becomes insensitive to the choice of  $\bar{\mu}$  as long as  $\bar{\mu} = O(m_h)$ . This has been checked in Ref. [12] for four-loop evolution in connection with three-loop matching. Armed with our new results, we are in a position to explore the situation at the next order. As an example, we study how the  $\bar{\mu}$  dependence of the relationship between  $\alpha_s^{(4)}(M_\tau)$  and  $\alpha_s^{(5)}(M_Z)$  is reduced as we implement five-loop evolution with four-loop matching. We first calculate  $\alpha_s^{(4)}(\bar{\mu})$  with Eq. (4) by imposing the condition  $\alpha_s^{(4)}(M_\tau) = 0.34$  [1], then obtain  $\alpha_s^{(5)}(\bar{\mu})$  from the on-shell version of Eq. (10) with  $M_b = 4.85$  GeV [1], and finally compute  $\alpha_s^{(5)}(M_Z)$  with Eq. (4). In Fig. 2, the variation of  $\alpha_s^{(5)}(M_Z)$  with  $\bar{\mu}/M_b$  is displayed for the various levels of accuracy,

respectively. The sums with  $k < 5$  may be found in Ref. [17], while those with  $k = 5$  may be obtained from there through integration,

$$\sum_{l=1}^{\infty} \frac{f(l)x^l}{l^{c+1}} = \int_0^x \frac{dy}{y} \sum_{l=1}^{\infty} \frac{f(l)y^l}{l^c}, \quad (7)$$

where  $f(l) = S_{\pm a}(l), \dots$ . After some algebra, we find an analytic expression for  $J_0$  and hence also for  $X_0$ ,

$$X_0 = -318\zeta(4) \ln 2 + \frac{873}{2} \zeta(5) - 48a_5 + O(\epsilon). \quad (8)$$

Here and in the following we use the constants

$$\begin{aligned} a_4 &= \left( -2\zeta(2) + \frac{\ln^2 2}{3} \right) \ln^2 2 + 8\text{Li}_4\left(\frac{1}{2}\right), \\ a_5 &= \frac{1}{3} \left( 2\zeta(2) - \frac{\ln^2 2}{5} \right) \ln^3 2 + 8\text{Li}_5\left(\frac{1}{2}\right). \end{aligned} \quad (9)$$

If we measure the matching scale  $\bar{\mu}$  in units of the  $\overline{\text{MS}}$  mass  $m_h(\bar{\mu})$ , our result for the ratio of  $a' = a^{(n_l)}(\bar{\mu})$  to  $a = a^{(n_l)}(\bar{\mu})$  reads

ranging from one-loop to five-loop evolution. While the

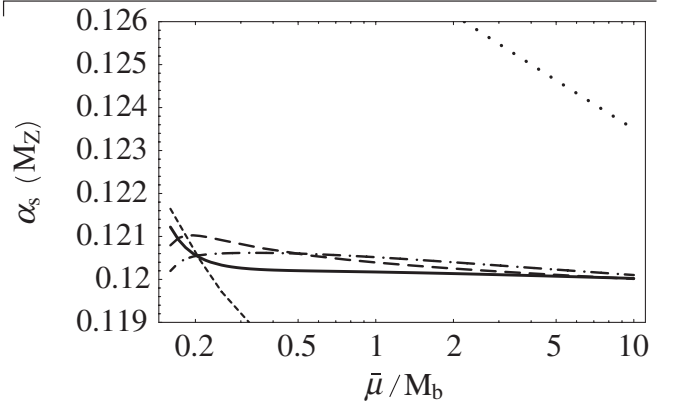


FIG. 2.  $\bar{\mu}$  dependence of  $\alpha_s^{(5)}(M_Z)$  from  $N$ -loop evolution and  $(N-1)$ -loop matching, with  $N = 1$  (dotted line), 2 (short-dashed line), 3 (dot-dashed line), 4 (long-dashed line), and 5 (solid line).

leading-order result exhibits a strong logarithmic behavior, the analysis is gradually getting more stable as we go to higher orders. The five-loop curve is almost flat. Besides the  $\bar{\mu}$  dependence of  $\alpha_s^{(5)}(M_Z)$ , also its absolute normalization is significantly affected by the higher orders.

As we have learned from Fig. 2, in higher orders, the actual value of  $\bar{\mu}$  does not matter as long as it is

comparable to the heavy-quark mass. In the context of Eq. (10), the choice  $\bar{\mu} = \mu_h$ , where  $\mu_h = m_h(\mu_h)$  is the renormalization-group (RG) invariant  $\overline{\text{MS}}$  mass, is particularly convenient, since it eliminates the RG logarithm  $\ell$ . With this convention, we obtain from Eqs. (3), (4), and (10) a simple relationship between  $\Lambda' = \Lambda^{(n_f)}$  and  $\Lambda = \Lambda^{(n_f)}$ ,

$$\begin{aligned} \beta'_0 \ln \frac{\Lambda'^2}{\Lambda^2} = & (\beta'_0 - \beta_0)l + (b'_1 - b_1)\ln l - b'_1 \ln \frac{\beta'_0}{\beta_0} + \frac{1}{\beta_0 l} [b_1(b'_1 - b_1)\ln l + b_1^2 - b_1^2 - b'_2 + b_2 + c_2] \\ & + \frac{1}{(\beta_0 l)^2} \left[ \frac{b_1^3}{2}(\ln^2 l - 1) - b'_1 b_1^2 \left( \frac{1}{2} \ln^2 l - \ln l - 1 \right) - b_1(b_1^2 - b'_2 + b_2 + c_2)\ln l - \frac{b_1^3}{2} + b'_1(b'_2 - b_2 - c_2) \right. \\ & \left. - \frac{1}{2}(b'_3 - b_3) + c_3 \right] + \frac{1}{(\beta_0 l)^3} \left\{ -b_1^4 \left( \frac{1}{3} \ln^3 l - \frac{1}{2} \ln^2 l - \ln l - \frac{1}{6} \right) + b'_1 b_1^3 \left( \frac{1}{3} \ln^3 l - \frac{3}{2} \ln^2 l - \ln l + \frac{1}{2} \right) + b_1^2 \right. \\ & \times (b_1^2 - b'_2 + b_2 + c_2)(\ln^2 l - \ln l - 1) + b_1 [b_1^3 - 2b'_1(b'_2 - b_2 - c_2) + b'_3 - b_3 - 2c_3] \ln l + \frac{b_1^4}{3} \\ & \left. - b_1^2(b'_2 - b_2 - c_2) + (b'_2 - b_2) \left( \frac{b'_2}{3} - \frac{2}{3} b_2 - c_2 \right) - c_2^2 + b'_1 \left( \frac{2}{3} b'_3 - \frac{b_3}{2} - c_3 \right) - \frac{b_1 b_3}{6} - \frac{1}{3}(b'_4 - b_4) + c_4 \right\} + O\left(\frac{1}{l^4}\right), \end{aligned} \quad (12)$$

where  $l = \ln(\mu_h^2/\Lambda^2)$ . The  $O(1/l^3)$  term of Eq. (12) is new. Equation (12) represents a closed four-loop formula for  $\Lambda^{(n_f)}$  in terms of  $\Lambda^{(n_f)}$  and  $\mu_h$ . For consistency, it should be used in connection with the five-loop expression (4) for  $\alpha_s^{(n_f)}(\mu)$  with the understanding that the underlying flavor thresholds are fixed at  $\bar{\mu} = \mu_h$ . The inverse relation that gives  $\Lambda^{(n_f)}$  as a function of  $\Lambda^{(n_f)}$  and  $\mu_h$  emerges from Eq. (12) via the substitutions  $\Lambda \leftrightarrow \Lambda'$ ;  $\beta_N \leftrightarrow \beta'_N$  for  $N = 0, \dots, 4$ ; and  $c_N \rightarrow -c_N$  for  $N = 2, 3, 4$ .

In conclusion, we extended the standard description of the strong-coupling constant in the  $\overline{\text{MS}}$  renormalization scheme to include five-loop evolution and four-loop matching at the flavor thresholds. These results are indispensable to relate the QCD predictions for different observables at next-to-next-to-next-to-next-to-leading order.

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