Chaos without Nonlinear Dynamics

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A linear, second-order filter driven by randomly polarized pulses is shown to generate a waveform that is chaotic under time reversal. That is, the filter output exhibits determinism and a positive Lyapunov exponent when viewed backward in time. The filter is demonstrated experimentally using a passive electronic circuit, and the resulting waveform exhibits a Lorenz-like butterfly structure. This phenomenon suggests that chaos may be connected to physical theories whose underlying framework is not that of a traditional deterministic nonlinear dynamical system.

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Chaos is a behavior that is usually attributed to nonlinear dynamical systems. Yet surprisingly, it has recently been shown that continuous-time chaotic waveforms can also be constructed by the linear superposition of certain pulse basis functions [1–3]. An important attribute of these basis pulses is an infinitely long, exponentially increasing precursor that provides determinism and a positive Lyapunov exponent. This linear synthesis of chaos may have practical application in the understanding and analysis of chaotic systems; for example, synthesized chaotic waveforms may be used to generate surrogate data with specified characteristics to test time series analysis and parameter estimation algorithms [3]. However, it is unclear if this mechanism can explain the presence of chaos in a physical system.

In this Letter, we show that the basis pulses are easily realized when viewed backward in time, and we use this observation to develop a remarkably simple physical system for generating chaotic waveforms. In particular, we find that a linear, second-order filter driven by a random bipolar signal can generate a waveform that, when viewed under time reversal, exhibits the essential qualities of chaos, including determinism and a positive Lyapunov exponent. We call such dynamics *reverse-time chaos*. We demonstrate its physical viability by implementing the filter in an electronic circuit using only a few passive linear components, and we obtain a waveform that exhibits a Lorenz-like butterfly structure.

Previously, it has been shown that a linear filter can increase the apparent dimensionality of a chaotic signal [4]. It is also known that a random source can emulate the statistical properties of a deterministic system [5]. But it is surprising that a linear filter driven by a purely random process produces a waveform that appears equally to have been produced by a deterministic chaotic system. This phenomenon suggests that chaos may be connected to physical theories whose underlying framework is not that of a traditional deterministic nonlinear dynamical system.

The synthesis of reverse-time chaos is markedly different from integrating a chaotic flow backwards in time. For such flows, the chaotic set is not an attracting set in reverse time. In contrast, the passive linear filter we construct for synthesizing reverse-time chaos is globally stable.

Currently, we are unaware of reverse-time chaos in any natural system; however, the simplicity of the filter mechanism implies one must allow for the possibility of it occurring naturally. Consequently, interpreting measurements of a physical system may require the arrow of time to distinguish deterministic chaos from simple filtering of random processes. Powerful chaos-detection algorithms that rely only on geometric analysis of state space structures, such as fractal dimension, template analysis [6], or false nearest neighbors [7], may not detect a difference between these sources. However, algorithms that incorporate the direction of time can differentiate conventional forward-time chaos due to a nonlinear dynamical system and reverse-time chaos from a filtered random process.

In addition, there are potential technology applications for reverse-time chaos. For communications, symbolic dynamics can be encoded directly into a reverse-time chaotic transmitter by modulating the polarity of the basis pulses [8]. At the receiver, the determinism of reverse-time chaotic waveforms provides a form of intentional intersymbol interference that may be processed using simple predictive filters [2,9]. For correlation-based ranging using chaotic waveforms, it is noted that reverse-time waveforms can work equally well [10–13]. Reverse-time chaos may also be relevant to the prediction and control of electromagnetic interference in circuits driven by radio frequency signals [14,15].

We begin by considering the chaotic shift map

$$z_{n+1} = 2z_n \bmod 1 \tag{1}$$

for an initial condition $0 \leq z_0 < 1$. If the state z_n is written as a binary fraction, the map corresponds to a left shift followed by dropping the integer bit. For example, the initial condition $z_0 = 0.110\,101\,11$, which is known with limited (truncated) precision, maps to $z_1 = 0.101\,011\,1?$. The ? indicates a bit of new information that was previously unknown due to limited knowledge of the initial condition. Subsequent iterations continue to reveal new bits of information, and this apparent capability for a deterministic map to generate new information explains chaos's extreme sensitivity to initial conditions and positive entropy.

To describe reverse-time chaos, we consider the inverse shift map

$$y_{n+1} = \frac{y_n + \sigma_n}{2} \tag{2}$$

for an initial condition $0 \le y_0 < 1$, where $\sigma_n \in \{0, 1\}$ is a random sequence. In binary representation, the map (2) defines a right shift followed by inserting σ_n as the new most significant bit. For example, the initial condition $y_0 = 0.110\,101\,11$ maps to $y_1 = 0.?11\,010\,11$, where the ? indicates the new bit of information supplied by σ_1. Thus, the iterated inverse map acts opposite to chaos by absorbing information from a random source. Since precise knowledge of the current state defines all prior iterates, the output of the map is deterministic when viewed in reverse time, and the backward iterates satisfy the chaotic shift map (1). Thus the inverse map (2), which is effectively a linear filter driven by a random process, generates reverse-time chaos.

We now extend the construction of reverse-time chaos to a continuous-time system. To this end, we consider the driven second-order linear system

$$\ddot{x} + 2\beta\dot{x} + (\omega^2 + \beta^2)x = s(t), \tag{3}$$

where $x(t)$ is the scalar dependent state, $\beta > 0$ is the decay rate, and $\omega > 0$ is the frequency of the damped oscillations. The input forcing $s(t)$ is

$$s(t) = As_n, \qquad n \le t < n + 1, \tag{4}$$

where each $s_n \in \{-1, +1\}$ is random and A is the drive amplitude. Since Eq. (3) is linear, we set $A = 1$ without loss of generality. The homogeneous solution is

$$x_h(t) = Ce^{-\beta t}\cos(\omega t + \phi), \tag{5}$$

where C and ϕ are integration constants. To obtain a shift dynamic, we seek homogenous solutions that satisfy

$$x_h(t + 1) = \tfrac{1}{2}x_h(t) \tag{6}$$

for all t. A nontrivial solution requires $\beta = \ln 2$ and $\omega = 2k\pi$, where $k \in \{1, 2, \ldots\}$. For these values, we show in the following that the linear system (3) driven by the random signal (4) generates reverse-time chaos.

To solve the general system, we consider the response of the linear system (3) to excitation by a single pulse of unit duration. That is, we solve the initial value problem

$$\ddot{\xi} + 2\beta\dot{\xi} + (\omega^2 + \beta^2)\xi = \begin{cases} 1, & 0 \le t < 1, \\ 0, & t \ge 1, \end{cases} \tag{7}$$

subject to the homogenous initial conditions $\xi(0) = \dot{\xi}(0) = 0$. For $\beta = \ln 2$ and $\omega = 2k\pi$, we find

$$\xi(t) = \frac{1}{\omega^2 + \beta^2}\begin{cases} 1 - e^{-\beta t}\left[\cos(\omega t) + \frac{\beta}{\omega}\sin(\omega t)\right], & 0 \le t < 1, \\ e^{-\beta t}\left[\cos(\omega t) + \frac{\beta}{\omega}\sin(\omega t)\right], & t \ge 1. \end{cases} \tag{8}$$

The pulse responses for $k = 1$, $k = 2$, and $k = 3$ are plotted in Fig. 1. We extend each solution to negative time by defining $\xi(t) = 0$ for $t < 0$. A general solution to Eq. (3) is then found by superposition of the unit pulse response,

$$x(t) = \sum_{n=-\infty}^{[t]} s_n \xi(t - n), \tag{9}$$

where $[t]$ indicates the largest integer less than or equal to t. Using Eq. (8), we obtain

$$x(t) = \frac{s_{[t]} + 2^{[t]-t}[\cos(\omega t) + \frac{\beta}{\omega}\sin(\omega t)]\{-s_{[t]} + \sum_{i=1}^{\infty} 2^{-i}s_{[t]-i}\}}{\omega^2 + \beta^2}. \tag{10}$$

We claim that the waveform (10) is chaotic in reverse time. This claim can be justified using the results of Hirata and Judd [3], who derived necessary conditions for the basis pulse function to assure the superposed dynamics are conjugate to a chaotic shift map. Instead, here we use a more direct approach by explicitly showing that a shift dynamics exists in the solution (10). To this end, we define a Poincaré return at integer times $t = [t]$. The nth return is then

$$x(n) = \frac{1}{\omega^2 + \beta^2}\sum_{i=1}^{\infty} 2^{-i}s_{n-i}. \tag{11}$$

Defining the scaled return

$$y_n = \frac{\omega^2 + \beta^2}{2}x(n) + \frac{1}{2} \tag{12}$$

yields

$$y_n = \sum_{i=1}^{\infty} 2^{-i}\sigma_{n-i}, \tag{13}$$

where $\sigma_n = \frac{s_n+1}{2}$ maps the bipolar symbols $s_n \in \{-1, +1\}$ to the bits $\sigma_n \in \{0, 1\}$. It is easy to verify that successive scaled returns (13) satisfy the inverse map (2); therefore, the returns also satisfy a chaotic shift map when viewed in reverse time. Consequently, the continuous-time waveform (10) also exhibits chaos in reverse time. Note that this does not imply that integrating Eq. (3) in reverse time will generate a chaotic waveform. Instead, this result implies that the solution of Eq. (3) will, when viewed in reverse time, exhibit the properties of a "chaotic" waveform,

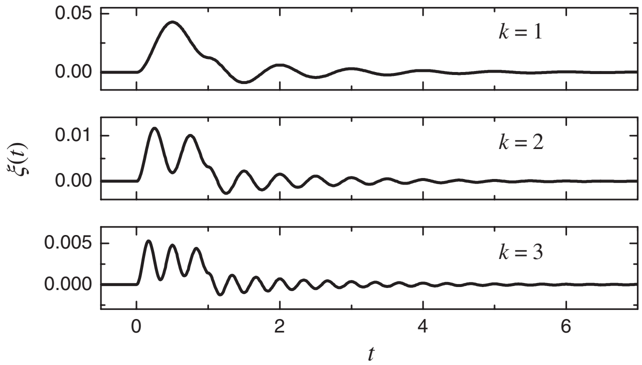

FIG. 1. Unit pulse response for $k = 1$, $k = 2$, and $k = 3$.

namely, determinism (the shift map) and a positive Lyapunov exponent ($\beta = \ln 2$).

In Fig. 2(a), we show a solution obtained by integrating Eq. (3) with $k = 1$ for a random sequence $s(t)$. The output waveform, when viewed backward in time, is visually similar to chaotic waveforms produced by the Lorenz system [16]. In Fig. 3(a) we show an "attractor" constructed using delay embedding of the waveform with $\Delta t = \frac{1}{3}$. Technically, the geometric structure shown in Fig. 3(a) is not an attractor, since a flow is not defined for states off the structure [2]; however, there is a topological similarity of the representation in Fig. 3(a) to the butterfly attractor of the Lorenz system. Further, chaotic waveforms from both systems are characterized by a continuous power spectrum and positive entropy, although the entropy of the reverse-time waveform is set externally by the characteristics of the random source while that of the Lorenz system is derived from its natural invariant density. The reverse-time waveform also differs from a Lorenz waveform in that it has a regular return time and a closed form analytic expression.

In the linear synthesis of chaos, the basis pulse explicitly contains information on the chaotic waveform's Lyapunov

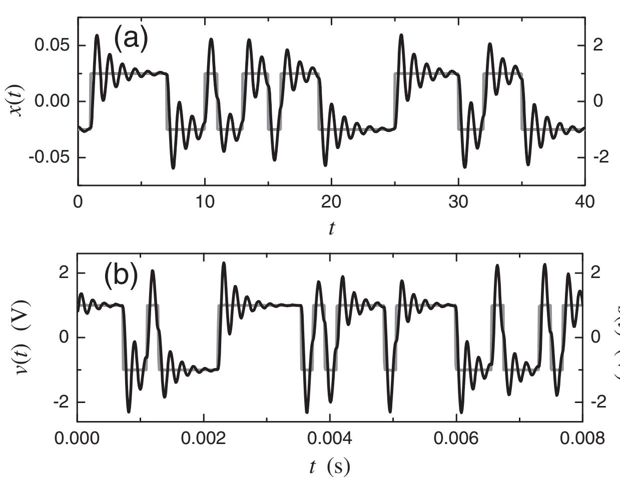

FIG. 2. Reverse-time chaotic waveform (a) generated using Eq. (3) with $k = 1$ and (b) measured in the electronic filter circuit. The random drive sequences are shown in gray.

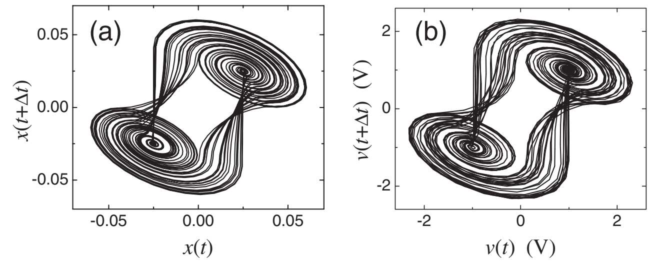

FIG. 3. Attractor constructed using time-delay embedding for waveform (a) generated using Eq. (3) with $k = 1$ ($\Delta t = 1/3$) and (b) measured in the filter circuit ($\Delta t = 60\ \mu$s).

exponents in both the expanding and contracting directions [2]. Adapted for reverse-time chaos, the positive exponent is the decay rate ($\beta = \ln 2$) for the tail of the output basis pulses shown in Fig. 1. This exponent governs the separation rate of generic reverse-time chaos waveforms from nearby initial conditions. Conversely, the negative exponent for the reverse-time waveform derives from the basis pulse rise time. However, since the rise time is finite for a causal filter, the negative exponent for physically realizable reverse-time chaos must be infinite. Thus, the reverse-time waveform exhibits complete dissipation in finite time and is not invertible; i.e., its prior states are not uniquely defined by its current state. In addition, the waveform's embedded attractor has no thickness and is described by an integer dimension. In contrast, dissipation in the Lorenz system is characterized by a finite negative exponent; thus, the Lorenz waveform is invertible and its attractor has a fractal dimension with Cantor-set thickness in the transverse direction.

Significantly, Eq. (3) models a number of physical systems including, for example, a damped linear pendulum. Here we demonstrate reverse-time chaos using the electronic filter shown in Fig. 4. Among the simplest circuits known to students, this circuit is modeled as

$$LC\frac{d^2v}{dt^2} + RC\frac{dv}{dt} + v = v_{\rm in}(t), \tag{14}$$

where $v(t)$ is the voltage across the capacitor. The applied drive voltage $v_{\rm in}(t)$ is

$$v_{\rm in}(t) = V_{\rm in}s_n, \qquad n \le \frac{t}{T} < n + 1, \tag{15}$$

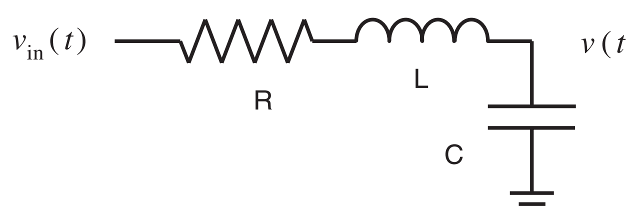

FIG. 4. Filter circuit consisting of a series resistance (R), inductance (L), and capacitance (C) and used for generating reverse-time chaos.

where $V_{\rm in}$ is a fixed amplitude, $s_n \in \{-1, +1\}$ is a random sequence, and T is the fundamental period of the drive waveform. Introducing $\tau = \frac{t}{T}$ yields

$$\frac{d^2 v}{d\tau^2} + \frac{TR}{L}\frac{dv}{d\tau} + \frac{T^2}{LC} v = f(\tau), \tag{16}$$

where

$$f(\tau) = \frac{T^2 V_{\rm in}}{LC} s_n, \qquad n \leq \tau < n + 1. \tag{17}$$

We note that the system (16) and (17) is in the same form as system (3) and (4); thus, the circuit can exhibit reverse-time chaos for parameters satisfying $\frac{TR}{L} = 2\ln 2$ and $\frac{T^2}{LC} = 4k^2\pi^2 + (\ln 2)^2$, where $k \in \{1, 2, \ldots\}$.

We implement the circuit shown in Fig. 4 using discrete electronic components. We use an inductor with $L = 7.5$ mH and intrinsic series resistance $R = 57\ \Omega$. For $k = 1$, the requirements for reverse-time chaos give $T = 180\ \mu$s and $C = 0.11\ \mu$F. To drive the circuit, we use a digital signal processor card (Innovative Integration ADC64) hosted in a personal computer to generate random ± 1 V bipolar bits at 5.6 kHz. The output waveform $v(t)$ is sampled at 100 kHz using a 12-bit data acquisition card (Keithley DAS1802).

A typical output waveform captured from the filter is shown in Fig. 2(b). In Fig. 3(b) we show the measured attractor constructed by delay embedding with $\Delta t = 60\ \mu$s. Again, we note the similarity of the reverse-time waveform and attractor to those of the chaotic Lorenz system.

The realization of reverse-time chaos provides a complementary view of determinism and randomness in nature. Forward-time chaos produces information by amplifying microscopic details of the initial state that were beyond the observer's ability to measure [17]. Even though the unfolding information is new to the observer, the future dynamics are completely determined by the present state. Reverse-time chaos turns this determinism around: the present state stores its entire history. Seeing further into the past requires greater measurement precision, implying a sensitive dependence on initial conditions and a positive Lyapunov exponent in reverse time. In this way, the random process generating reverse-time chaos is entirely consistent with a deterministic process generating forward-time chaos.

The filter developed here generates reverse-time chaos when driven by a random binary signal. Such waveforms are ubiquitous, for example, in computer and data communication systems. However, it is natural to consider extensions of the filter construction to other random sources. Using random square pulses with more than two levels is possible but requires larger dissipation in the filter to maintain reverse-time determinism; consequently, the reverse-time waveform has a larger positive Lyapunov exponent consistent with the increased entropy of the random source. It is also interesting to consider irregular pulse timing and continuous noise, although entropy arguments suggest some quantization may be necessary. We hope to develop such extensions in future research.

In this Letter, we have shown that reverse-time chaos appears in a physical system comprising a linear filter driven by a random process. We believe it is surprising that chaos in any form can be generated by such a remarkably simple system. That the language of deterministic chaos provides a useful description for signals not generated by a nonlinear dynamical system suggests chaos may be more fundamental than previously supposed. At the very least, the possibility of chaos by such a simple mechanism provides a new perspective on the interplay of deterministic and random processes in nature.

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