Collisions of Strings with Y Junctions

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We study the dynamics of Nambu-Goto strings with junctions at which three strings meet. In particular, we exhibit one simple exact solution and examine the process of intercommuting of two straight strings in which they exchange partners but become joined by a third string. We show that there are important kinematical constraints on this process. The exchange cannot occur if the strings meet with very large relative velocity. This may have important implications for the evolution of cosmic superstring networks and non-Abelian string networks.

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There has been renewed interest in cosmic strings, both because of tentative observational evidence [1,2] and because they appear to arise naturally in scenarios based on string theory [3–5], as well as in field theories [6,7]. Although the recent close inspection by the Hubble Space Telescope of the area of interest in Refs. [1,2] appears to indicate that no string is present [8], the possibility remains that strings may be found through other types of observation in the Universe. Moreover, in the scenarios based on string theory, several different kinds of cosmic strings may appear, in particular, F- and D-strings and (p, q) composites, formed of p F-strings and q D-strings [9]. In such cases, junctions may form at which three different strings meet. Such junctions can also appear in networks of ''non-Abelian strings,'' for which the fundamental group $\pi_1(\mathcal{M})$ of the manifold of degenerate vacua, which classifies the strings, is non-Abelian [10,11]. Several papers have considered the evolution of networks of strings with junctions [12–15], in particular, the question of whether such a network would evolve to a scaling regime as expected for an ordinary cosmic-string network [16].

In this Letter, we study the dynamics of three-string junctions in a local-string network, that is to say, one where the individual strings have no long-range interactions and are well described by the Nambu-Goto action. Our approach is similar to the one adopted by 't Hooft in Ref. [17], in which he represented baryons as pieces of open string connected at one common point. However, our method differs from his in significant ways. In particular, our treatment applies to strings with different tensions, and we use a temporal world-sheet coordinate equal to the global time. For ordinary cosmic strings, the existence of exact solutions for oscillating string loops [18,19] was important in analyzing the likely behavior of loops in general, and some exact solutions are also known for open strings with junctions. Here we give one very simple example of an exact solution, but our main focus is on the question of what happens when two strings cross.

When two ordinary cosmic strings intersect, they normally ''intercommute,'' or exchange partners [20–24]. But for the strings we are considering, this is generally impossible. What we expect instead is that the strings will become joined by a third string. The dynamical problem of finding the intercommuting probability for junction-forming strings has been discussed by several authors [25,26]. Our study, of the interaction of a pair of straight strings, is complementary. We will show that there are important *kinematical* constraints implying that such intercommuting is impossible for strings that meet with very high relative velocity. The limit depends on the angle at which the strings meet and on the ratios of the string tensions. (This does not appear to be related to the bound found in simulations in Ref. [26], which has a quite different dependence on the angle.) Although colliding (p, q) strings generally have different tensions, we consider explicitly here only the case in which the two initial strings have equal tension—though the third string that joins them may have a different tension—because the inherent symmetry of the problem makes this relatively easy to solve. However, similar limits apply more generally, as we will show in a later publication.

We first review the equations of motion of strings with junctions. We use the standard conformal gauge conditions, in which the temporal world-sheet coordinate is identified with the time $\tau = t$, and the spatial coordinates $\mathbf{x}(\sigma, t)$ satisfy the gauge conditions

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$$\dot{\mathbf{x}} \cdot \mathbf{x}' = 0, \qquad \dot{\mathbf{x}}^2 + \mathbf{x}'^2 = 1, \tag{1}$$

where $\dot{\mathbf{x}} = \partial_t \mathbf{x}$ and $\mathbf{x}' = \partial_\sigma \mathbf{x}$.

Let us consider a junction of three strings, with coordinates $\mathbf{x}_j(\sigma, t)$ and string tensions μ_j ($j = 1, 2, 3$). (We take σ to be increasing towards the vertex on all three strings.) The position of the vertex is denoted by $\mathbf{X}$ and the values of the spatial world-sheet coordinates σ there by $s_j(t)$. Then the action may be written

$$S = -\sum_j \mu_j \int dt \int d\sigma \theta(s_j(t) - \sigma) \sqrt{\mathbf{x}_j'^2 (1 - \dot{\mathbf{x}}_j^2)} + \sum_j \int dt \mathbf{f}_j(t) \cdot [\mathbf{x}_j(s_j(t), t) - \mathbf{X}(t)], \tag{2}$$

where the $\mathbf{f}_j$ are Lagrange multipliers.

Varying $\mathbf{x}_j$ and using (1) yields the usual equation of motion

$$\ddot{\mathbf{x}}_j - \mathbf{x}_j'' = \mathbf{0}, \tag{3}$$

but there are also boundary terms proportional to $\delta(s_j(t) - \sigma)$ which give

$$\mu_j(\mathbf{x}_j' + \dot{s}_j \dot{\mathbf{x}}_j) = \mathbf{f}_j, \tag{4}$$

where the functions are evaluated at $(s_j(t), t)$. Varying the Lagrange multipliers $\mathbf{f}_j$, of course, provides the boundary conditions

$$\mathbf{x}_j(s_j(t), t) = \mathbf{X}(t), \tag{5}$$

while varying $\mathbf{X}$ provides the constraint

$$\sum_j \mathbf{f}_j = \mathbf{0}. \tag{6}$$

Finally, varying s_j and using (1) again, we get

$$\mathbf{f}_j \cdot \mathbf{x}_j' = \mu_j \mathbf{x}_j'^2, \tag{7}$$

which is not an independent equation but an immediate consequence of (4).

The general solution of (3) is, of course,

$$\mathbf{x}_j(\sigma, t) = \tfrac{1}{2}[\mathbf{a}_j(\sigma + t) + \mathbf{b}_j(\sigma - t)], \tag{8}$$

where to satisfy the gauge condition (1) we require

$$\mathbf{a}_j'^2 = \mathbf{b}_j'^2 = 1. \tag{9}$$

Thus, (5) becomes

$$\mathbf{a}_j(s_j + t) + \mathbf{b}_j(s_j - t) = 2\mathbf{X}(t). \tag{10}$$

In addition, from (4) and (6) we have

$$\sum_j \mu_j[(1 + \dot{s}_j)\mathbf{a}_j' + (1 - \dot{s}_j)\mathbf{b}_j'] = \mathbf{0}. \tag{11}$$

The initial conditions for $\mathbf{x}_j$ and $\dot{\mathbf{x}}_j$ at $t = 0$ serve to fix the functions $\mathbf{a}_j(\sigma)$ and $\mathbf{b}_j(\sigma)$ for $\sigma < s_j(0)$. (There will also be lower limits on the ranges of σ, determined by boundary conditions at the other ends of the strings, but for the moment we assume that they are far enough away to be irrelevant.) In the subsequent motion, the amplitudes of the inward-moving waves at the vertex, namely, $\mathbf{b}_j'(s_j - t)$, will thus be known, but the amplitudes of the outgoing waves $\mathbf{a}_j'(s_j + t)$ will not. They can be found from the various junction conditions as follows.

First, differentiating (10), we find [at $\sigma = s_j(t)$]

$$(1 + \dot{s}_j)\mathbf{a}_j' - (1 - \dot{s}_j)\mathbf{b}_j' = 2\dot{\mathbf{X}}. \tag{12}$$

Substituting for $\mathbf{a}_j'$ from this equation into (11), we get

$$\sum_j \mu_j(1 - \dot{s}_j)\mathbf{b}_j' = -(\mu_1 + \mu_2 + \mu_3)\dot{\mathbf{X}}. \tag{13}$$

Eliminating $\dot{\mathbf{X}}$ from (12) and (13), each $\mathbf{a}_j'$ can be expressed as a linear combination of the $\mathbf{b}_j'$. We still have to determine the unknown values of $\dot{s}_j$, but this can be done by imposing the gauge conditions $\mathbf{a}_j'^2 = 1$. The result is a function of the string tensions μ_j, as well as the scalar products

$$c_{ij} = \mathbf{b}_i'(s_i - t) \cdot \mathbf{b}_j'(s_j - t) = c_{ji}. \tag{14}$$

It turns out to be simplest to solve for the three unknowns $1 - \dot{s}_j$. To express the result concisely, we define three combinations M_j of the string tensions by

$$M_1 = \mu_1^2 - (\mu_2 - \mu_3)^2, \tag{15}$$

and two similar equations for M_2 and M_3. Then we find

$$\frac{\mu_1(1 - \dot{s}_1)}{\mu_1 + \mu_2 + \mu_3} = \frac{M_1(1 - c_{23})}{M_1(1 - c_{23}) + M_2(1 - c_{31}) + M_3(1 - c_{12})}, \tag{16}$$

together with two similar equations obtained by cyclic permutation.

There is one important immediate corollary, which follows from the obvious restriction $\dot{s}_j \leq 1$. This implies that each $M_j \geq 0$. In other words, the three string tensions must satisfy the triangle inequalities; if one tension exceeds the sum of the other two, no three-string junction is possible. This is, of course, obvious for the case of a static equilibrium configuration.

Note that summing the three equations (16) yields the relation

$$\mu_1 \dot{s}_1 + \mu_2 \dot{s}_2 + \mu_3 \dot{s}_3 = 0. \tag{17}$$

This is an expression of energy conservation: The rate of creation of new string must balance the disappearance of old.

Equations (16) serve to determine the values of $s_j(t)$. Note that these are differential equations for s_j rather than an explicit solution, because in the light of the definition (14) the values of $s_j(t)$ also appear on the right-hand side. So, in general, a numerical solution may be needed. Once

we have found the $s_j(t)$, we can at once write down the values of $\mathbf{a}_j'(s_j + t)$ from (12) and (13) and then integrate to find $\mathbf{a}_j$.

Of course, this process can proceed only so long as the relevant values of $\mathbf{b}_j'$ are within the range determined by the initial conditions. Eventually, the effects of other junctions will come into play, and the values of $\mathbf{b}_j'$ will be ones determined by earlier dynamics at these other junctions, not by the initial conditions. Nevertheless, an iterative solution of the dynamical equations for all the junctions together is, in principle, feasible.

As an example, it is easy to generalize the familiar collapsing circular loop solution to a configuration comprising three semicircular arcs, namely,

$$\mathbf{x}_j(\sigma, t) = \cos t(\cos\sigma\cos\theta_j, \cos\sigma\sin\theta_j, \sin\sigma), \tag{18}$$

where $|\sigma| \leq \pi/2$, and the angles θ_j are chosen to satisfy the equilibrium conditions

$$\sum_j \mu_j e^{i\theta_j} = 0. \tag{19}$$

This is always possible provided the μ_j satisfy the triangle inequalities. Clearly, here we can take $\mathbf{a}_j(\sigma) = \mathbf{b}_j(\sigma) = \mathbf{x}_j(\sigma, 0)$. From (14) and (19), it is then straightforward to verify that, when each $s_j = \pi/2$, (16) implies that all $\dot{s}_j = 0$, so (18) is a self-consistent solution. The loops remain semicircular and shrink to a point at time $t = \pi/2$.

We now turn to our central problem: What happens when two strings that can exchange partners, becoming linked by a third string, meet? In general, this can happen in two different ways, and it is not obvious which one is chosen, or indeed whether they do exchange partners at all.

Consider two straight strings approaching one another along the z axis. For simplicity, we discuss in this Letter the case of equal tension $\mu_1 = \mu_2$, although similar results hold more generally (as we shall describe in a future publication). For $t < 0$, we take

$$\mathbf{x}_{1,2}(\sigma, t) = (-\gamma^{-1}\sigma\cos\alpha, \mp\gamma^{-1}\sigma\sin\alpha, \pm vt). \tag{20}$$

Here v is the string velocity and $\gamma^{-1} = \sqrt{1 - v^2}$. Thus,

$$\begin{aligned} \mathbf{a}_{1,2}' &= (-\gamma^{-1}\cos\alpha, \mp\gamma^{-1}\sin\alpha, \pm v), \\ \mathbf{b}_{1,2}' &= (-\gamma^{-1}\cos\alpha, \mp\gamma^{-1}\sin\alpha, \mp v). \end{aligned} \tag{21}$$

(The sign of σ is chosen to match our earlier conventions.)

After the strings cross, we expect that they will be joined by a third string, which by symmetry must lie either along the x axis or along the y axis. *A priori*, we might guess that if the angle α is small, the connecting string would be in the x direction, while if it is closer to $\pi/2$, it would choose the y direction. We shall see that this is partly correct.

To be specific, let us suppose that the connecting string (labeled 3) is along the x axis. (See Fig. 1.) Then clearly, in this region,

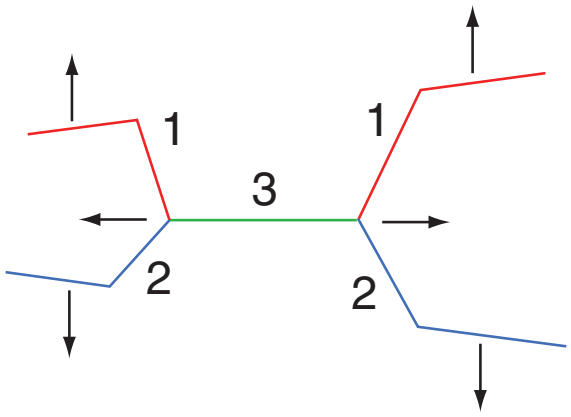

FIG. 1 (color online). Two strings joined by a third after intercommuting.

$$\mathbf{x}_3(\sigma, t) = (\sigma, 0, 0), \qquad \mathbf{a}_3'(\sigma) = \mathbf{b}_3'(\sigma) = (1, 0, 0). \tag{22}$$

There will be two junctions at the ends of this string, symmetrically placed on either side of the origin. By the symmetry of the problem, it is sufficient to consider one of them; we shall take the one on the positive x axis. Clearly, using (17), $s_1 = s_2 = -(\mu_3/2\mu_1)s_3$. The vertex position is $\mathbf{X}(t) = (s_3(t), 0, 0)$. It moves along the x axis with uniform velocity $\dot{s}_3$.

To apply the previous method, we first evaluate the c_{ij} of (14) and then substitute into (16). This yields

$$\dot{s}_3 = \frac{2\mu_1\gamma^{-1}\cos\alpha - \mu_3}{2\mu_1 - \mu_3\gamma^{-1}\cos\alpha}, \qquad \dot{s}_1 = \dot{s}_2 = -\frac{\mu_3}{2\mu_1}\dot{s}_3. \tag{23}$$

On the strings 1 and 2, for $t > 0$ there are kinks at $\sigma = t$, at the positions $t(\gamma^{-1}\cos\alpha, \pm\gamma^{-1}\sin\alpha, \pm v)$, beyond which the expressions (20) still apply. The vertex $\mathbf{X}$ is joined to these kinks by new straight segments. From (12), one can find values of $-\mathbf{b}_{1,2}'$ and $\mathbf{a}_3'$ representing the outgoing waves and verify that they are consistent.

This result has interesting implications. It is clear that the solution makes sense only if $\dot{s}_3 > 0$: the connecting string 3 cannot get *shorter*. Thus, we require

$$\alpha < \arccos\left(\frac{\mu_3\gamma}{2\mu_1}\right) \qquad (x \text{ axis}). \tag{24}$$

This is in line with our expectation that the connecting string would form on the x axis for small α. For a string along the y axis, one would require

$$\alpha > \arcsin\left(\frac{\mu_3\gamma}{2\mu_1}\right) \qquad (y \text{ axis}). \tag{25}$$

As we have already noted, no junction is possible if $\mu_3 > 2\mu_1$. Moreover, for any mass ratio there is a limiting velocity above which a junction cannot form; we require

$$\gamma < \frac{2\mu_1}{\mu_3}. \tag{26}$$

For example, if the tensions are all equal, no junction can form unless $v < \sqrt{3}/2$. Strings approaching each other at very large velocity *cannot* exchange partners. Abelian strings will simply pass through one another. For non-Abelian strings, if this is topologically forbidden, they may become joined by a new string along the z axis, but *without* exchanging partners. It turns out that this process is kinematically allowed provided that

$$v > \frac{\mu_3}{2\mu_1} \qquad (z \text{ axis}), \tag{27}$$

independent of α. Note that if μ_3 is large there may be a range of velocities for which neither inequality (26) nor (27) can be satisfied. In that case, the strings must be locked into an X configuration, unable to separate in any direction. This may have important implications for the evolution of a cosmological network of such strings.

It is worth noting that there are cases in which two or more of these configurations are possible. In such cases, the choice may be random.

The discussion can be extended to the collision of two strings of different tension $\mu_1 \neq \mu_2$, with similar results, although in general we have not found an analytic formula for the limiting velocity. We shall discuss the general case in a later paper.

We have shown that there are important kinematic constraints on the possibility of intercommuting of strings that form junctions. If the relative velocity with which they meet is too large, no exchange can take place and the strings will merely pass through one another (or, for non-Abelian strings, become joined by a string in the direction of the relative velocity or form a linked X configuration). This restriction, and the difference in behavior between non-Abelian strings and junction-forming Abelian strings, may be of considerable importance in studies of the evolution of a cosmological string network. Of course, even if the kinematic constraints are satisfied, there is no guarantee that intercommuting will occur. This dynamical problem has been discussed by several authors [25,26] with the conclusion that the intercommuting probability is frequently much less than 1. The effect of reducing the intercommuting probability on the evolution of a network of standard cosmic strings (with no junctions) has been considered in Refs. [27–30].

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