Long-Distance Entanglement in Spin Systems

L. Campos Venuti,[1,2] C. Degli Esposti Boschi,[1,3] and M. Roncaglia[1,2,3]
[1]*Dipartimento di Fisica, Università di Bologna, Viale C. Berti-Pichat 6/2, I-40127 Bologna, Italy*
[2]*INFN Sezione di Bologna, Viale C. Berti-Pichat 6/2, I-40127 Bologna, Italy*
[3]*CNR-INFM, Unità di Bologna, Viale C. Berti-Pichat 6/2, I-40127 Bologna, Italy*
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Most quantum system with short-ranged interactions show a fast decay of entanglement with the distance. In this Letter, we focus on the peculiarity of some systems to distribute entanglement between distant parties. Even in realistic models, like the spin-1 Heisenberg chain, sizable entanglement is present between arbitrarily distant particles. We show that long-distance entanglement appears for values of the microscopic parameters which do not coincide with known quantum critical points, hence signaling a transition detected only by genuine quantum correlations.

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Entanglement generation and distribution is a problem of central importance in performing quantum-information (QI) tasks, like teleportation [1] and quantum cryptography [2]. Typically, the entanglement between parties is created by means of a direct interaction. Since entanglement needs the presence of strong correlations, low-dimensional systems, as, for example, antiferromagnetic spin chains, offer a natural source of entanglement. In most systems with short-range interactions, the entanglement between a pair of particles decays rapidly with the distance (generally even more rapidly than standard correlations). For example, in the Ising model with transverse field [3] the concurrence vanishes for distances larger than 2 sites, while in the Heisenberg model [4] it is restricted only to nearest neighbors.

From the QI perspective, it would be attracting to create sizable entanglement between particles that are located at a distance larger than a few sites. Along this direction, the localizable entanglement was conceived with the idea of exploiting spin chains as quantum channels [5]. The localizable entanglement measures the average entanglement localized between a couple of distant points, after performing optimal local measurements onto the rest.

In this Letter, we show that already the ground state (GS) of various models widely used in condensed matter physics offer the possibility to entangle parties that are arbitrarily far apart. This fact naturally leads to the concept of long-distance entanglement (LDE) as a sort of quantum order parameter. As discussed in the following, the onset of LDE does not coincide with known quantum phase transitions (QPT's) of the systems we have examined.

Let us consider two sites A and B that interact with a many-body system C. The distance d between A and B is set by the individual short-ranged interactions in the subsystem C (see Fig. 1).

According to our definition, given a bipartite measure of entanglement $E(\rho)$, we have LDE if

$$E_d(\rho_{\mathrm{AB}}) \xrightarrow{d\to\infty} E_\infty \neq 0$$

where $\rho_{\mathrm{AB}} = \mathrm{Tr}_{\mathrm{C}} |\Psi\rangle\langle\Psi|$ is the reduced density matrix of the subsystem A and B, and $|\Psi\rangle$ is the total wave function. The introduction of two special points, or probes, is essential here since the property of monogamy [6] limits to two the number of particles maximally entangled. The basic idea comes from the observation that if we wish to locate a great amount of entanglement between two selected qubits, we are forced to exclude entanglement with the rest. Specifically, we have considered cases where A and B represent end spins in an open chain or additional spins (probes) that interact with selected sites in the chain. In condensed matter systems, these might be impurities, defects, or even scattering particles [7].

As a first criterion, we expect to have a nonvanishing LDE between A and B when their interactions with C are *small* compared to the typical interactions contained in C. Otherwise, A or B would develop too strong correlations with the closest degrees of freedom in C, excluding the possibility to form LDE. On the other side, strong correlations among the particles in C tend to avoid entanglement between C and the probes. In this sense, strongly correlated quantum systems, like antiferromagnetic spin systems, are good candidates to do the job. In particular, spin-$1/2$ antiferromagnetic systems admit a simple picture based on resonating valence bonds (RVBs) [8]. If a state is a total spin singlet, then it may be approximated by all the possible RVB configurations, each one with a given weight. Resonances between various configurations destroy entanglement. The variational idea for favoring a singlet between two selected sites (A and B) is to induce a large weight for all the RVB configurations that link pairs of particles inside C by increasing here the interactions.

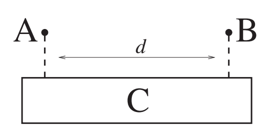

FIG. 1. A schematic setup for creating entanglement between two selected sites A and B at distance d, through the subsystem C.

In the following, we present some mechanisms able to produce LDE in spin-1/2 and spin-1 chains.

The dimerized-frustrated model.—In the $S = 1/2$ antiferromagnetic isotropic Heisenberg chain, each spin is highly entangled with its nearest neighbors [9]. Instead we consider here the dimerized chain with frustration, well-known for its connections with spin-Peierls [10] and ladder compounds, whose Hamiltonian is

$$\mathcal{H} = \sum_{j=1}^{L-1}[1 + \delta(-1)^j]\vec{\sigma}_j \cdot \vec{\sigma}_{j+1} + \alpha \sum_{j=1}^{L-2} \vec{\sigma}_j \cdot \vec{\sigma}_{j+2} \quad (1)$$

where σ^ν, $\nu = x, y, z$ are Pauli matrices. For $\delta = 0$ the system is gapless up to $\alpha_c \approx 0.241$, where the GS spontaneously dimerizes and becomes doubly degenerate. In the Majumdar-Ghosh line $\delta + 2\alpha = 1$ the system is made only by short-ranged singlets.

We choose L even and open boundary conditions (OBCs), with the aim to study the entanglement between the two spin-1/2 at the end points. First, let us look at two limit cases. For $\delta = -1$ and $\alpha = 0$ the GS is dimerized onto the "odd" bonds,

where the entanglement is localized in pairs of nearest neighbors. More interesting for us is the case $\delta = 1$ and $\alpha = 0$, where two spins are "left alone" as in the following figure

and the GS is fourfold degenerate. The basic idea for concentrating a large amount of entanglement between end spins (A and B) is related to their tendency to form a global singlet in the GS for any $\delta \neq 1$. Hence the two end states are forced to develop strong correlations towards the formation of a long-distance singlet state $|\Psi^-\rangle \equiv (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)/\sqrt{2}$. This phenomenon can be thought of as a long-range antiferromagnetic interaction mediated by the other spins in the chain. Accordingly, the states in $S_{\rm tot} = 1$ form a triplet of excitations.

As a measure of entanglement, we adopt the concurrence [11]. Given the SU(2) invariance of the GS, the spin-spin correlations $\gamma_{ij}^{\nu\nu} \equiv \langle \sigma_i^\nu \sigma_j^\nu \rangle/4$ are the same for every ν. In addition, the magnetization is zero, so that the concurrence between A and B reduces simply to $\mathcal{C}_{\rm AB} = 2\max\{0, 2|\gamma_{\rm AB}^{zz}| - \gamma_{\rm AB}^{zz} - 1/4\}$. The concurrence is nonzero if the antiferromagnetic correlations between A and B are sufficiently strong: $\gamma_{\rm AB}^{zz} < -1/12$.

First, we have performed some numerical evaluations on the GS using the density matrix renormalization group (DMRG) method [12]. The end-to-end concurrence $\mathcal{C}_{\rm AB}$ is plotted in Fig. 2 as a function of the system size L for several values of δ and α. The numerical data put in evidence the presence of LDE as well as the rapid achievement of the asymptotic value. This latter feature is consistent with the small correlation length in the regime

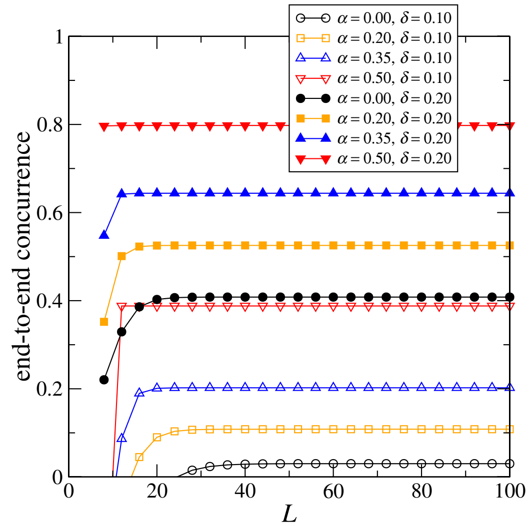

FIG. 2 (color online). The finite-size study on the concurrence shows the presence of LDE in model (1) for $\delta > \delta_T(\alpha)$. Data were obtained keeping 256 DMRG states, with a truncation error smaller than 10^{-10}.

$\delta \gtrsim 0.10$ (see, e.g., Ref. [10]) and allows us to study shorter chains by means of an exact diagonalization program based on the Lanczos method.

The numerical results summarized in Fig. 3 show that the end-to-end concurrence grows rapidly with δ starting from a threshold value $\delta_T(\alpha)$. For $\delta = 0$ no LDE is generated, and this is related to the tendency of the first spin to entangle with the second, as found in Ref. [13]. This is also consistent with the absence of surface order in an open $S = 1/2$ Heisenberg chain [14].

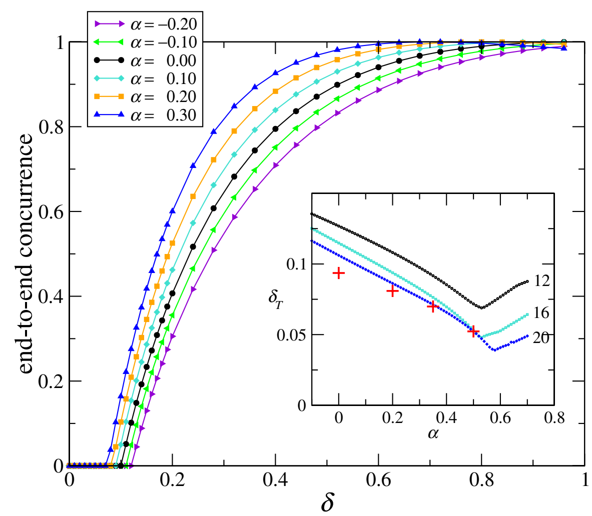

FIG. 3 (color online). Concurrence for the two end-qubit state as a function of dimerization δ for some values of α. Exact calculation on a chain of length $L = 24$. The LDE increases steeply above a threshold and is enhanced by frustration. Inset: threshold value of dimerization $\delta_T(\alpha)$, above which end-to-end concurrence starts to be nonzero for lengths $L = 12, 16, 20$. The crosses are the infinite size extrapolations of DMRG data with L up to 100.

The inclusion of $\alpha < 0$ tends to favor a classical Néel state, so it is expected to destroy entanglement. On the contrary, $\alpha > 0$ is seen to enhance the end-qubit concurrence, as frustration favors quantum fluctuations. This is shown in the inset of Fig. 3 where $\delta_T(\alpha)$ decreases with α, reaching a minimum for $\alpha \approx 0.5$. In the limit $|\alpha| \gg 1$ the entanglement gets suppressed as the probes belong to two separated chains. Remarkably, from Figs. 2 and 3 it emerges that the entanglement grows with the system size L.

Spin-1 chain.—An important class of spin-1 models is given by the Heisenberg chain with biquadratic interactions,

$$\mathcal{H} = \sum_{i=1}^{L-1} [\vec{S}_i \cdot \vec{S}_{i+1} + \beta(\vec{S}_i \cdot \vec{S}_{i+1})^2],$$

that has attracted much interest both for the study of hidden order [15] and for optical lattice implementations [16]. At the Affleck-Kennedy-Lieb-Tasaki (AKLT) point $\beta = 1/3$, the GS is given by a valence bond solid (VBS) [15], where each spin 1 is represented by a couple of spin-1/2, provided the antisymmetric state is projected out. The VBS state is constructed by forming short-ranged singlets between nearest neighbor $S = 1/2$ states and then symmetrizing local pairs to get back $S = 1$ states. For OBC there remains free effective spin-1/2 particles at the endpoints responsible for a fourfold degeneracy, in an analogous way as in the model (1) with $\delta = 1$ and $\alpha = 0$. Away from $\beta = 1/3$ the degeneracy is lifted, the GS is a total singlet $S_{\text{tot}} = 0$, and other valence bond configurations give contribution to the GS. Anyway, the VBS state is still a good approximation for a wide range of β's, in particular, at the Heisenberg point $\beta = 0$. Because of strong correlations in the bulk, the two $S = 1/2$ end spins tend to organize as a $|\Psi^-\rangle$ Bell state, in order to give rise to a total singlet.

Two different measures were considered to quantify the entanglement between two spin-1. The VBS picture suggests the definition of the partial concurrence (PC) as the amount of entanglement between the spin-1/2 belonging to different spin-1 particles. One advantage of the PC is that it depends only on the z-z correlator: $\text{PC}_{\text{A,B}} = 2\max\{0, 2|\eta^{zz}_{\text{AB}}| - \eta^{zz}_{\text{AB}} - 1/4\}$, with $\eta^{zz}_{\text{AB}} = \langle S^z_{\text{A}} S^z_{\text{B}} \rangle/4$. The symmetrization procedure distributes the entanglement among the four qubit, so that the maximal possible value of the PC is 1/2. However, the PC may fail in detecting genuine qutrit entanglement, which is generally hard to quantify. In fact, for qutrit mixed states there is no simple expression for the entanglement of formation nor a simple criterion for separability is known. Nevertheless, for SU(2)-rotationally invariant states, a necessary and sufficient condition for a state to be entangled is that of having positive negativity [17], defined as

$$\mathcal{N}(\rho_{\text{AB}}) = \|\rho_{\text{AB}}^{T_{\text{A}}}\|_1 - 1,$$

where $\rho_{\text{AB}}^{T_{\text{A}}}$ stands for the partial transpose with respect to subsystem A and $\|G\|_1 = \text{tr}\sqrt{GG^\dagger}$. Calculating the negativity of a general SU(2)-invariant state, parametrized by the quantities $\langle S^z_{\text{A}} S^z_{\text{B}} \rangle$ and $\langle (S^z_{\text{A}})^2 (S^z_{\text{B}})^2 \rangle$, we are able to recognize the separable states. All the possible states fall inside the triangle in Fig. 4, whereas the shaded area represents the separable states.

In the AKLT case $\beta = 1/3$, we choose the singlet among the four degenerate states, because this is the state one would approach by letting $\beta \rightarrow 1/3$ and corresponds to the GS of the periodic chain. From the exact solution, one finds $\langle S^z_1 S^z_L \rangle = -\langle (S^z_1)^2 (S^z_L)^2 \rangle \simeq -4/9[1 + 6(-1)^L e^{-L/\xi_{\text{AKLT}}}]$, where $\xi_{\text{AKLT}} = 1/\ln(3)$ is the bulk AKLT correlation length. It follows that in the thermodynamic limit PC $= 1/6$ and $\mathcal{N} = 2/9$, where both values are approached exponentially fast. This confirms the hypothesis that we have qubit as well as qutrit entanglement.

At the Heisenberg point $\beta = 0$ with OBC it is well established the presence of surface order $\lim_{L\to\infty} \langle S^z_1 S^z_L \rangle = -0.283\,064\,84(1)$ [18] approached also in this case exponentially fast with a bulk correlation length $\xi_H \sim 6$. With accurate DMRG simulation up to 100 sites we could establish a similar behavior for the correlations $\langle (S^z_1)^2 \times (S^z_L)^2 \rangle$ with asymptotic value very close to 4/9. These data imply the existence of LDE in the Heisenberg model detected by a nonzero negativity $\mathcal{N} = 0.060\,842\,6$, even if qubit entanglement vanishes, i.e., PC $= 0$. We note here that both the Heisenberg (H) and the AKLT points (see Fig. 4) lie on the line where $\langle (S^z_1)^2 (S^z_L)^2 \rangle \xrightarrow{L\to\infty} \langle (S^z_1)^2 \rangle \times \langle (S^z_L)^2 \rangle = 4/9$, which means that the nonzero spins (effective charges) are uncorrelated. Further enhancement of the LDE may be achieved in spin-1 models that present also end-to-end charge correlations.

$S = 1/2$ *Heisenberg model with probes.*—So far, we have considered situations where the probes are located at the end points of a chain. Now we consider a different case: a Heisenberg chain of length L and two additional $S = 1/2$ probes $\vec{\tau}_{\text{A}}$ and $\vec{\tau}_{\text{B}}$

$$\mathcal{H} = \sum_{j=1}^{L} \vec{\sigma}_j \cdot \vec{\sigma}_{j+1} + J_p(\vec{\sigma}_1 \cdot \vec{\tau}_{\text{A}} + \vec{\sigma}_{d+1} \cdot \vec{\tau}_{\text{B}})$$

where $\vec{\tau}$ is a vector of Pauli matrices. The spin-probe A interacts with the site 1, while B is connected to the site $d + 1$. The correlations between A and B will depend only by their distance $d + 2$, having assumed periodic boundary conditions with $\sigma^\alpha_{L+1} \equiv \sigma^\alpha_1$.

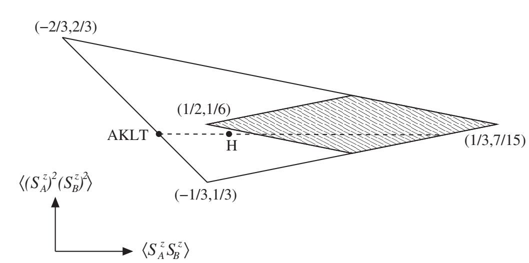

FIG. 4. Entangled (white) and separable (shaded) states of two globally SU(2)-invariant qutrits are completely determined by means of $\langle S^z_{\text{A}} S^z_{\text{B}} \rangle$ and $\langle (S^z_{\text{A}})^2 (S^z_{\text{B}})^2 \rangle$.

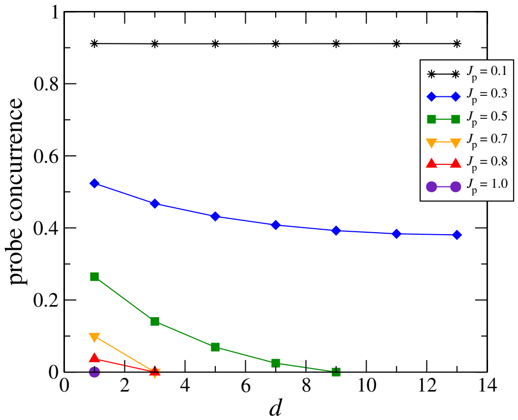

FIG. 5 (color online). Concurrence between probes attached to a Heisenberg chain of length $L = 26$ as a function of the distance. The calculation was done for various values of J_p. A dramatic increase of entanglement between distant probes appears as J_p is lowered.

By means of exact diagonalization, we have studied the concurrence $\mathcal{C}_{\mathrm{AB}}(d)$, varying the distance at fixed L. The results are illustrated in Fig. 5 for a closed chain of $L = 26$ (plus 2 probes). We have rejected even values of d, as in these cases the GS is threefold degenerate, belonging to the sector $S_{\mathrm{tot}} = 1$. Moreover, due to the periodic boundary conditions the maximum distance is reached at half chain, $d = 13$. When $J_p = 1$, no probe entanglement is found, since $\mathcal{C}_{\mathrm{AB}}(d) = 0$ for every d. As we expected according to our considerations above, $\mathcal{C}_{\mathrm{AB}}$ is enhanced by weakening the interactions between the probes and the spin chain. Already for $J_p = 0.3$, the entanglement is nonzero for every (odd) value of d and remarkably at $J_p = 0.1$ the probes are almost completely entangled. Finite-size scaling of the concurrence between maximally distant probes, $\mathcal{C}_{\mathrm{AB}}(d = L/2, L)$, exhibits a slow decrease of the concurrence with L and it remains an open question whether it survives at the thermodynamic limit.

In addition, we just mention that changing the sign of the probe interactions to ferromagnetic, $J_p < 0$, the concurrence increases further, extending the possibility of tailoring interactions that yield efficient entanglement creation at large distance. A similar behavior is observed by placing the probes at the ends of an open chain. As above, the finite-size effects are non-negligible due to the critical nature of the bulk. Preliminary DMRG calculations with L up to 100 leave open the possibility of having LDE for ferromagnetic probe interactions.

Conclusions.—With this Letter, we aim to bring to the attention of the QI community a large class of spin-1/2 and spin-1 models capable of creating entanglement between distant parties. On the one hand, this property opens up the possibility to engineer QI devices like entanglers and quantum channels using strongly correlated low-dimensional systems. In particular, the phenomenon of concentrating the entanglement on the border of finite-size system seems to be particularly suited for optical lattice simulations. On the other hand, we observe that the transition point where genuinely quantum correlations, signaled by the concurrence, extend to long distance does not coincide with known QPT's. How this issue embodies in the statistical mechanics framework is a challenging question. Conversely, local measures of entanglement show a singular behavior at QPT's that comes from the most relevant operator [19]. Specifically, in this work we have considered models with SU(2) symmetry, which is common in nature and help to make the calculations easier. Nonetheless, we verified that the results regarding LDE apply also to non SU(2)-symmetric cases. Further work is in progress in order to extend the investigation of LDE on other models.

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