Understanding the $\frac{5}{2}$ Fractional Quantum Hall Effect without the Pfaffian Wave Function

Csaba Tőke and Jainendra K. Jain

Department of Physics, 104 Davey Laboratory, The Pennsylvania State University, Pennsylvania 16802, USA

(Received 18 April 2006; published 22 June 2006)

It is demonstrated that an understanding of the $\frac{5}{2}$ fractional quantum Hall effect can be achieved within the composite fermion theory without appealing to the Pfaffian wave function. The residual interaction between composite fermions plays a crucial role in establishing incompressibility at this filling factor. This approach has the advantage of being amenable to systematic perturbative improvements, and produces ground as well as excited states. It, however, does not relate to non-Abelian statistics in any obvious manner.

DOI: 10.1103/PhysRevLett.96.246805 PACS numbers: 73.43.−f, 71.10.Pm

The $\frac{5}{2}$ fractional quantum Hall effect (FQHE) [1,2] has received renewed attention of late because of the intriguing possibility of its utilization in quantum computation. The currently most promising picture for this FQHE is in terms of "the Pfaffian state," proposed by Moore and Read [3] and Greiter, Wen, and Wilczek [4,5]. The Pfaffian wave function describes a real-space p-wave-paired BCS wave function for a fixed the number of composite fermions (CFs). The best evidence in favor of the Pfaffian state comes from numerical studies, which have shown [6–8] that for small systems in the second Landau level (LL), the Pfaffian wave function has a reasonably good overlap with the exact Coulomb ground state. The Pfaffian also is the exact ground state for a model three-body short-range interaction, $\mathcal{V}_3$ [Eq. (2)]. Exact solutions for this model are also available for quasiholes, which have been shown, theoretically, to constitute a realization of "non-Abelian anyons" [3,9–11].

This picture, however, is not entirely satisfactory. It is not known how the Pfaffian wave function, which does not contain any variational parameters, can be improved for the two body Coulomb interaction. The pairing of composite fermions is viewed as arising from an instability of the CF Fermi sea [4,5,12], but the CF Fermi sea is not a limiting case of the Pfaffian wave function. No satisfactory quantitative understanding currently exists for the excitations of the $\frac{5}{2}$ state; we see evidence below that the three-body interaction $\mathcal{V}_3$ does not capture the qualitative physics of the actual excitations of the Coulomb $\frac{5}{2}$ state. Finally, the actual meaning of pairing of composite fermions remains unclear; the $\frac{5}{2}$ state has no off-diagonal long-range order, and, in an operational sense, it does not appear different from other FQHE states.

These considerations have motivated us to seek another approach for describing the physics of the $\frac{5}{2}$ FQHE, on which we elaborate in this Letter. We still proceed within the CF framework, but without assuming any pairing at the outset. No FQHE occurs at $\nu = \frac{5}{2}$ in a model that neglects interactions between composite fermions, which predicts many degenerate ground states at this filling factor. [This is to be contrasted with the FQHE at $\nu = n/(2pn \pm 1)$.] We show below that the residual interaction between composite fermions opens a gap to produce an incompressible state. Furthermore, the results can be improved systematically within a perturbative scheme. This approach produces ground state as well as low-lying excitations.

Below, the lowest LL is assumed to be full and inert; only the half-filled second LL is considered explicitly, and full spin polarization of electrons is assumed. Lengths are measured in the units of the magnetic length, $l_B = \sqrt{\hbar c/eB}$, and energies in units of $e^2/\epsilon l_B$ (ϵ is the dielectric constant of the host semiconductor). The spherical geometry [13] will be employed, which considers N electrons moving on the surface of a sphere with a magnetic monopole of strength Q at the center, producing a magnetic flux of strength $2Q\phi_0$, where $\phi_0 = hc/e$ is the magnetic flux quantum. The total orbital angular momentum quantum number is denoted by L.

The Pfaffian state assumes the form

$$\Psi^{\mathrm{Pf}} = \mathrm{Pf}\left(\frac{1}{u_i v_j - v_i u_j}\right)\Phi_1^2, \quad \Phi_1 = \prod_{i<j}(u_i v_j - v_i u_j), \quad (1)$$

where $u_i = \cos\frac{\theta_i}{2}e^{-i\phi_i/2}$ and $v_i = \sin\frac{\theta_i}{2}e^{i\phi_i/2}$. It is the exact ground state of the short-range three-body interaction [4,7,10]

$$\mathcal{V}_3 = V\sum_{i<j<k} P_{ijk}(L_{\max}), \quad (2)$$

where $P_{ijk}(L_{\max})$ is the projection operator onto an electron triplet with orbital angular momentum $L_{\max} = 3Q - 3$. $\mathcal{V}_3$ penalizes configurations with electron triplets in their closest configuration.

The CF theory [14] describes the two-dimensional electron system in terms of composite fermions, which are electrons bound to an even number (taken to be two in this Letter) of quantized vortices. The lowest LL splits into "λ levels" of composite fermions, which are analogous to Landau levels of electrons in a reduced field $B^* = B - 2\rho\phi_0$. Microscopically, the CF formation is defined by the expression

$$\Psi_Q = P_{\mathrm{LLL}}\Phi_1^2\Phi_{Q^*}, \quad (3)$$

where Φ is a wave function for N electrons at monopole strength Q^*, the Jastrow factor Φ_1^2 attaches two vortices to them, and $P_{\rm LLL}$ projects the wave function into the lowest LL [15]. The monopole strengths are related by $Q = Q^* + N - 1$.

A technical obstacle toward a quantitative study of the state at $\nu = \frac{5}{2}$, defined here through the relation $2Q = 2N - 3$, is that composite fermions experience a negative magnetic field here, given by $Q^* = -\frac{1}{2}$. While the CF theory is known to be valid for negative B^* [16,17], the convenient projection method developed in Ref. [15] does not apply to such situations for technical reasons. [The recent work of Möller and Simon [17] can be useful in this respect, but we have not explored that.] We avoid negative values of Q^* by exploiting particle-hole symmetry to study $N_h = (2Q + 1) - N = N - 2$ *holes* at $2Q = 2N - 3$. Composite fermions *made from holes* experience a positive monopole strength

$$Q^* = Q - (N_h - 1) = 3/2. \tag{4}$$

The hole version of $\Psi^{\rm Pf}$ is found conveniently from its second-quantized form.

The single particle states at Q^* are monopole harmonics [18] Y_{Q^*lm}, where l is the angular momentum and m its z component. The LL index is given by $n = l - Q^*$ ($n \geq 0$). Independent many-fermion basis states Φ are Slater determinants of Y_{Q^*lm}'s at Q^*, specified by a set $\{l_i, m_i\}$. In the nth order of "CF diagonalization"[19], we collect all basis states with at most n units of CF kinetic energy above the minimum:

$$\{\{\Phi_\alpha^{(0)}\}, \{\Phi_\beta^{(1)}\}, \{\Phi_\gamma^{(2)}\}, \ldots, \{\Phi_\zeta^{(n)}\}\}.$$

A correlated CF basis at Q, of dimension $D^{(n)}$, is obtained through Eq. (3),

$$\{\{\Psi_\alpha^{(0)}\}, \{\Psi_\beta^{(1)}\}, \{\Psi_\gamma^{(2)}\}, \ldots, \{\Psi_\zeta^{(n)}\}\},$$

with $n_i = l_i - Q^*$ now interpreted as the λ-level index, and $\sum_i n_i$ as the total "CF kinetic energy." We diagonalize the Coulomb interaction V in this basis. That requires a Monte Carlo evaluation of the direct product and interaction matrices ($\langle\Psi_\alpha^{(n)}|\Psi_\beta^{(m)}\rangle$ and $\langle\Psi_\alpha^{(n)}|V|\Psi_\beta^{(m)}\rangle$, respectively), orthogonalization by the standard Gram-Schmidt procedure, and numerical diagonalization [19]. The ground state from the nth order CF diagonalization will be denoted by $\Psi_0^{(n)}$. The dimensions of various bases are given in Table I.

Monte Carlo CF diagonalization requires a real-space interaction. The Coulomb interaction of the second LL is simulated by an effective interaction in lowest LL of the form

$$V^{\rm eff}(r) = \frac{1}{r} + \sum_{i=0}^{M} c_i r^i, \tag{5}$$

where the coefficients c_i are fixed so that the lowest LL pseudopotentials [13] of $V^{\rm eff}(r)$ reproduce *all* of the second LL Coulomb pseudopotentials $V_m^{(1)}$ for odd integral values of m. The Coulomb pseudopotentials in the nth LL are given by

$$\begin{aligned} V_m^{(n)} = \frac{1}{R} \sum_{m1, m2 = -l}^{l} & \\ \times \sum_{j=|m_1-m_2|}^{2l} & (-1)^{j+m_2-m_1} \langle 2l - m, 0 | l, m_1; l, -m_1 \rangle \\ \times \langle 2l - m, 0 | l, m_2; l, -m_2 \rangle & | \langle l, m_1; j, m_2 - m_1 | l, m_2 \rangle \\ \times \langle l, Q; j, 0 | l, Q \rangle & |^2, \end{aligned} \tag{6}$$

where m is the relative angular momentum of two particles, $l = Q + n$, and $R = \sqrt{Q} l_B$ is the radius. [Equation (6) reduces to the expression in Fano *et al.* [20] for $n = 0$.] As Q depends on N, a distinct set of coefficients has to be calculated for each N. To fit Eq. (5) we use that the pseudopotentials of a monomial r^n in the lowest LL are ($J = 2l - m$)

$$\begin{aligned} V_m(r^n) = & \frac{2^{n+4}\pi^2}{(2Q + n/2 + 1)!(2J + 1)!} \\ & \times \sum_{k=0}^{J} \frac{(J!)^2 (J + k)! (2Q + n/2 - k)!}{k!(J - k)!} \frac{1}{R}. \end{aligned} \tag{7}$$

TABLE I. Dimensions of various bases for N_h particles at $2Q = 2N_h + 1$. $D_{\rm ex}$ is the size of the Hilbert space in the $L_z = 0$ sector, and $D^{(n)}$ is the dimension of the CF basis incorporating nth order λ-level mixing. $D_{L=0}^{(n)}$ is the number of CF states in the $L = 0$ sector. Asterisks mark the cases where we could not determine the number of linearly independent basis states.

N_h	$D_{\rm ex}$	$D^{(0)}$	$D_{L=0}^{(0)}$	$D^{(1)}$	$D_{L=0}^{(1)}$	$D^{(2)}$	$D_{L=0}^{(2)}$
6	151	3	1	14	2	42	3
8	1514	3	1	20	1	72	4
12	194 668	4	1	37	2	205	8
14	2 374 753	8	1	63	3	644*	18*
16	3×10^7	4	1	52	2	495*	14*
20	5×10^9	5	1	77	2	965*	18*

Figure 1 shows the excitation spectra for $N_h = 12$, 14, 16, and 20 obtained by CF diagonalization at the zeroth and the first orders. ($N_h = 18$ is not considered as it aliases with $\nu = 3/7$ of holes.) The residual interaction between composite fermions lifts the degeneracy between various states to produce an incompressible state already at the lowest (zeroth) order, which neglects λ-level mixing. Although the energy gaps change by up to 50% in going from the zeroth to the first order, the incompressibility is preserved, indicating that while λ-level mixing renormalizes composite fermions, it does not cause any phase transition. The overestimation of gaps at the zeroth order may be attributed to the very small dimensions of the CF basis. All CF basis states are perturbations of the noninteracting CF Fermi sea, making it explicit that a rearrangement of composite fermions near the CF Fermi level is responsible for the $\frac{5}{2}$ FQHE.

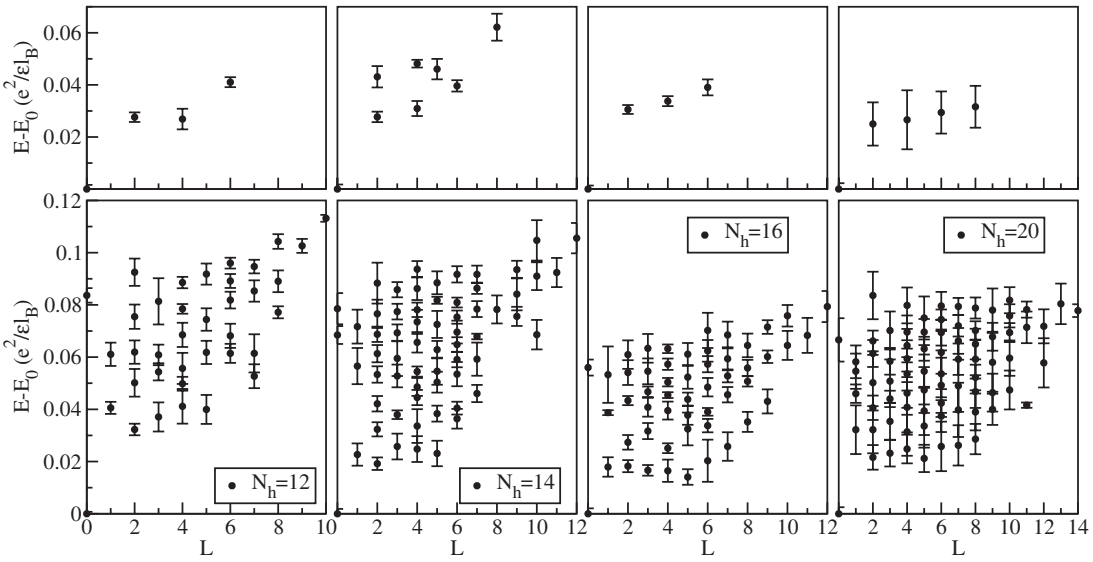

FIG. 1. Zeroth-order (top) and first-order (bottom) CF diagonalization excitation spectra for $N_h = 12, 14, 16, 20$.

When plotted as a function of $kl_B = L/\sqrt{Q}$, the lowest energy excitations for $14 \leq N_h \leq 20$ (from the first-order spectra) fall on a more or less continuous curve, which indicates that the thermodynamic behavior has been approached for $N_h \geq 14$. Finite-size effects are nonnegligible for $N_h < 14$. For $N_h > 20$ the first-order calculation is not sufficient, and the second-order CF diagonalization computationally too time consuming. Although there is some ambiguity as to which excitation is to be identified with the transport gap (corresponding to a far separated quasiparticle-quasihole pair), the existence of an almost flat region allows us to estimate a gap of ~ 0.02. This value is consistent with the earlier results from exact diagonalization [6,21].

Figure 2 shows a comparison of the CF spectra, at first- and second-order CF diagonalization, with the exact spectra for $N_h = 6$ and 8. The CF theory does not provide as accurate an account of the energies as it does for the lowest LL FQHE states. However, it works reasonably well for energy differences. The CF spectrum produces, at the first order, the energy gap to better than 25% accuracy. These comparisons thus provide credence to the semiquantitative validity of our approach.

Of interest also is the nature of multiquasihole states a few flux quanta away from $\nu = \frac{5}{2}$. Figure 3 shows spectra, for the $\mathcal{V}_3$ interaction, for $N = 10$ electrons at $2l = 18$ and $2l = 19$, which correspond to two and four "quasiholes" of the Pfaffian state. (We have switched back to electrons now, as these states occur at positive B^*.) This model predicts zero energy states at $L = 1, 3, 5$ and $L = 0^2, 1^0, 2^4, 3^1, 4^4, 5^2, 6^3, 7^1, 8^2, 9^0, 10^1$, respectively (the superscript denotes the degeneracy). No corresponding quasidegenerate band of states can be identified in the exact spectrum (middle columns). Figure 3 also shows spectra from first-order CF diagonalization. It produces a ground state at the correct quantum number but is not very successful for higher energy states. The CF spectrum can be improved systematically by incorporating higher order λ-level mixing.

The lack of a qualitative correspondence between the low-energy spectra of $\mathcal{V}_3$ and the Coulomb interactions in Fig. 3 raises questions regarding the validity of the $\mathcal{V}_3$ model, and hence of the model of quasiholes based on the Pfaffian wave function [9,10], for the *real* quasiholes of the $\frac{5}{2}$ state. This has relevance to the issue of statistics. Non-

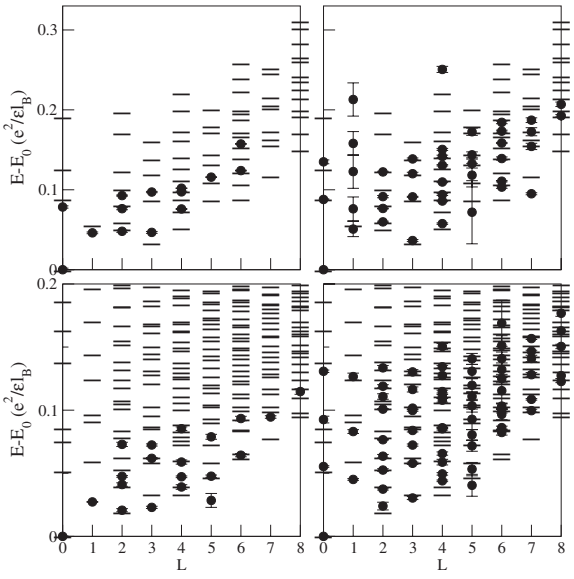

FIG. 2. First-order (left) and second-order (right) CF diagonalization excitation spectra for $N_h = 6$ (top) and $N_h = 8$ (bottom) holes. The dashes show the exact spectrum, and the dots the CF spectrum. The exact and the CF ground state energies for $N_h = 6$ are $E_{\rm ex}/N = -0.415\,217$, $E^{(1)}/N = -0.413\,609$, $E^{(2)}/N = -0.415\,233$; those for $N_h = 8$ are: $E_{\rm ex}/N = -0.401\,443$, $E^{(1)}/N = -0.395\,293$, $E^{(2)}/N = -0.399\,043$.

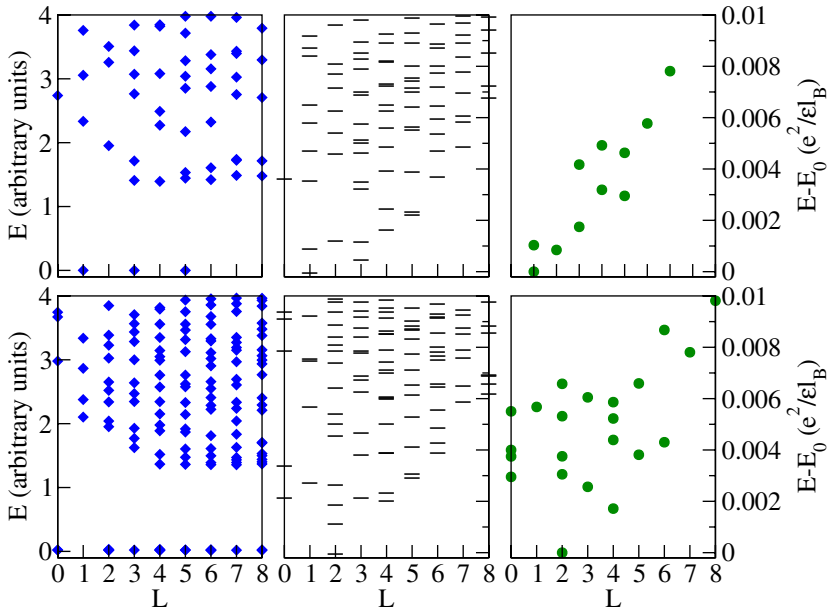

FIG. 3 (color online). Spectra at $\nu = 5/2$ for the model interaction $\mathcal{V}_3$ (left column), the Coulomb interaction (central column), and the first-order CF diagonalization (right column) for $N = 10$ particles at $2l = 18$ (top row) and $2l = 19$ (bottom row). For the $\mathcal{V}_3$ interaction, two (four) quasiholes are expected for $2l = 18$ ($2l = 19$). The ground state energies are $E_0/N = -0.415\,008$ (-0.40699) for exact and -0.40986 ($-0.401\,845$) for composite fermions, with $2l = 18$ ($2l = 19$). The energies in the middle column correspond to the scale shown on right. The spectra on the left were also given in Ref. [10].

Abelian statistics for the quasiholes of the $\mathcal{V}_3$ model is a consequence of the existence of several degenerate states for a given configuration of spatially localized quasiholes, which, in turn, is closely related to the degeneracy of the angular momentum eigenstates in Fig. 3. The spectra in Fig. 3 demonstrate a lack of adiabatic continuity, for the systems studied, between the many quasihole states of the $\mathcal{V}_3$ and the Coulomb models. (For the many quasiparticle or many quasihole states of the ordinary FQHE states in the lowest Landau level, the qualitative structure of a low-energy band predicted by the analogy to noninteracting *fermions* at Q^* is confirmed in similar exact spectra.)

In summary, we have demonstrated that the residual interaction between composite fermions causes incompressibility at $\nu = \frac{5}{2}$, and that the lowest order treatment of λ-level mixing gives a reasonable estimate of the activation energy. This model can be applied to neutral excitations at $\nu = \frac{5}{2}$ as well as charged excitations slightly away from $\nu = \frac{5}{2}$. The residual interaction may possibly induce pairing between composite fermions, but it is not known how to establish that, in a conclusive manner, within our approach.

The authors thank Chia-Chen Chang for the derivation of Eq. (7). We are grateful to him and N. Regnault for many useful comments and discussions. We thank the High Performance Computing (HPC) group at Penn State University ASET (Academic Services and Emerging Technologies) for assistance and computing time on the Lion-XL and Lion-XO clusters. Partial support of this research by the National Science Foundation under Grant No. DMR-0240458 is gratefully acknowledged.

[1] R. Willett, J. P. Eisenstein, H. L. Stormer, D. C. Tsui, A. C. Gossard, and J. H. English, Phys. Rev. Lett. **59**, 1776 (1987).
[2] W. Pan, J.-S. Xia, V. Shvarts, D. E. Adams, H. L. Stormer, D. C. Tsui, L. N. Pfeiffer, K. W. Baldwin, and K. W. West, Phys. Rev. Lett. **83**, 3530 (1999).
[3] G. Moore and N. Read, Nucl. Phys. **B360**, 362 (1991).
[4] M. Greiter, X. G. Wen, and F. Wilczek, Phys. Rev. Lett. **66**, 3205 (1991); Nucl. Phys. **B374**, 567 (1992).
[5] M. Greiter, X. G. Wen, and F. Wilczek, Phys. Rev. B **46**, 9586 (1992).
[6] R. H. Morf, Phys. Rev. Lett. **80**, 1505 (1998).
[7] E. H. Rezayi and F. D. M. Haldane, Phys. Rev. Lett. **84**, 4685 (2000).
[8] V. W. Scarola, J. K. Jain, and E. H. Rezayi, Phys. Rev. Lett. **88**, 216804 (2002).
[9] C. Nayak and F. Wilczek, Nucl. Phys. **B479**, 529 (1996).
[10] N. Read and E. Rezayi, Phys. Rev. B **54**, 16 864 (1996).
[11] Y. Tserkovnyak and S. H. Simon, Phys. Rev. Lett. **90**, 016802 (2003).
[12] V. W. Scarola, K. Park, and J. K. Jain, Nature (London) **406**, 863 (2000).
[13] F. D. M. Haldane, Phys. Rev. Lett. **51**, 605 (1983); *The Quantum Hall Effect*, edited by S. M. Girvin (Springer, New York, 1987).
[14] J. K. Jain, Phys. Rev. B **40**, 8079 (1989); **41**, 7653 (1990).
[15] J. K. Jain and R. K. Kamilla, Int. J. Mod. Phys. **B11**, 2621 (1997); Phys. Rev. B **55**, R4895 (1997).
[16] X. G. Wu, G. Dev, and J. K. Jain, Phys. Rev. Lett. **71**, 153 (1993).
[17] G. Möller and S. H. Simon, Phys. Rev. B **72**, 045344 (2005).
[18] T. T. Wu and C. N. Yang, Nucl. Phys. **B107**, 365 (1976).
[19] S. S. Mandal and J. K. Jain, Phys. Rev. B **66**, 155302 (2002).
[20] G. Fano, F. Ortolani, and E. Colombo, Phys. Rev. B **34**, 2670 (1986).
[21] R. H. Morf, N. d'Ambrumenil, and S. Das Sarma, Phys. Rev. B **66**, 075408 (2002).