

How Low-Energy Weak Reactions Can Constrain Three-Nucleon Forces and the Neutron-Neutron Scattering Length

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We show that chiral symmetry and gauge invariance enforce relations between the short-distance physics that occurs in a number of electroweak and pionic reactions on light nuclei. Within chiral perturbation theory, this is manifested via the appearance of the same axial isovector two-body contact term in $\pi^- d \rightarrow nn\gamma$, p -wave pion production in NN collisions, tritium β decay, pp fusion, νd scattering, and the hep reaction. Using a Gamow-Teller matrix element obtained from calculations of pp fusion as input, we compute the neutron spectrum obtained in $\pi^- d \rightarrow nn\gamma$. With the short-distance physics in this process controlled from $pp \rightarrow de^+\nu_e$, the theoretical uncertainty in the nn scattering length extracted from $\pi^- d \rightarrow nn\gamma$ is reduced by a factor larger than 3, to $\lesssim 0.05$ fm.

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In quantum chromodynamics (QCD), the up and down quark masses are much smaller than the typical scale of hadron masses $M_{\text{QCD}} \sim 1 \text{ GeV}/c^2$, and so the QCD Lagrangian has an approximate $SU(2)_L \times SU(2)_R$ symmetry. This symmetry is spontaneously broken by the QCD vacuum, leaving only $SU(2)_{\text{isospin}}$ as a manifest symmetry of low-energy hadronic processes. The spontaneously broken symmetry results in the existence of three (pseudo-)Nambu-Goldstone bosons, the pions, and means that the quantum-field-theoretic operator corresponding to the Noether currents of the broken generators changes pion number by one. External electroweak fields couple to these currents, and so the chiral $SU(2)_L \times SU(2)_R$ symmetry of QCD imposes relationships between the pion-baryon couplings and the axial couplings of baryons measured in electroweak processes. One of these is the Goldberger-Treiman relation:

$$\frac{g_A}{f_\pi} = \frac{g_{\pi NN}}{M}, \quad (1)$$

which expresses the πNN coupling constant $g_{\pi NN}$ in terms of the nucleon's axial coupling g_A and the pion decay constant f_π . (M is the nucleon mass.)

The symmetries of QCD have other consequences for low-energy reactions involving pions. Introducing a photon field into the theory according to the dictates of $U(1)_{\text{em}}$ gauge invariance shows that the leading piece of the amplitude for the photoproduction of π^+ or π^- at threshold is given by the Kroll-Ruderman (KR) term:

$$|\mathcal{A}_{\text{KR}}| = \frac{eg_A}{f_\pi}. \quad (2)$$

In the single-nucleon sector, the combination of $U(1)_{\text{em}}$ and QCD's spontaneously broken $SU(2)_L \times SU(2)_R$ enforces relations such as Eqs. (1) and (2). In this Letter, we explore the analog of these relations in the two-nucleon sector. We first exhibit a relationship between the short-distance physics in $NN \rightarrow NN\pi$ reactions and electroweak

processes such as pp fusion, tritium beta decay, and muon absorption on deuterium $\mu^- d \rightarrow nn\nu_\mu$. We then show that the same two-nucleon physics occurs in some reactions involving photons, in particular, in the process $\pi^- d \rightarrow nn\gamma$. The two-nucleon coupling in this reaction is then directly proportional to the two-nucleon coupling of the axial current. Knowledge of one coupling therefore yields the other, so that precise information about $\pi^- d \rightarrow nn\gamma$ (or its crossed partner $\gamma d \rightarrow nn\pi^+$) constrains pp fusion. Conversely, calculations of tritium beta decay and pp fusion contribute to model-independent predictions for $\pi^- d \rightarrow nn\gamma$.

Equations (1) and (2) are exact only in the chiral limit of QCD, i.e., when $m_u = m_d = 0$. In our world, they are satisfied to within 2% and 6%, respectively, since they have corrections proportional to m_u and m_d . The tool for systematically computing such corrections is chiral perturbation theory (χ PT) (see Ref. [1] for a review). χ PT is an effective field theory (EFT) in which amplitudes are obtained in an expansion in $\chi \equiv m_\pi/M_{\text{QCD}}$. Since its initial development [2,3], χ PT has been successfully applied to a number of processes in the meson-meson and meson-nucleon systems. But few-nucleon systems involve states which are bound by the strong interaction, and so the χ PT expansion must be modified. Weinberg [4] has pointed out that the χ PT expansion can be applied to the operator \hat{O} used in the computation of a matrix element

$$\mathcal{M} = \langle \psi_f | \hat{O} | \psi_i \rangle \quad (3)$$

for reactions involving such systems. The result—provided consistently computed wave functions $|\psi_{i,f}\rangle$ [5,6] are used—is an expansion of \mathcal{M} in powers of χ that incorporates the model-independent consequences of the (broken) chiral symmetry of QCD into calculations of processes in few-nucleon systems.

The operator \hat{O} in (3) is computed up to a given order in the χ PT expansion in powers of χ , using the Feynman

rules derived from the χ PT Lagrangian. At leading order, the relevant terms for one nucleon are

$$\mathcal{L}_N = N^\dagger [i v \cdot D + g_A S \cdot u] N, \quad (4)$$

where v is the nucleon four-velocity, D is a (chiral) covariant derivative, S is the Pauli-Lubanski spin vector, and g_A parametrizes the unknown short-distance physics of the nucleon's axial-current matrix element. Here u_μ is an axial four-vector which contains the pion field and when expanded takes the form:

$$f_\pi u_\mu = -\tau_a \partial_\mu \pi_a - \epsilon_{3ab} V_\mu \pi_a \tau_b + f_\pi A_\mu + \mathcal{O}(\pi^3), \quad (5)$$

where V_μ (A_μ) is an external vector (axial) field. The Goldberger-Treiman and KR relations (1) and (2) follow directly from inserting (5) into the Lagrangian (4) and examining the relevant πNN and $\gamma\pi NN$ couplings.

The axial four-vector u_μ also appears when pions, photons, and axial fields couple to pairs of nucleons. The leading effects of this can be represented by one term in the chiral Lagrangian [7,8]:

$$\mathcal{L}_{NN} = -2d_1 N^\dagger S \cdot u NN^\dagger N. \quad (6)$$

In principle, the low-energy constant (LEC) d_1 can be evaluated by lattice methods [9]. As with most single-nucleon sector LECs, however, its value has at present been obtained only from experimental data. Once the value of the LEC d_1 has been extracted from one process, we can use the result to predict other observables on which it has an impact. Equation (6) thus encodes model-independent correlations between different reactions which share the same short-distance physics because of the symmetries of the chiral Lagrangian. For the reactions discussed here, the short-distance NN physics is associated with ${}^3S_1 \leftrightarrow {}^1S_0$ transitions. It is convenient to parametrize it by a dimensionless constant \hat{d} , which is obtained from d_1 and from which the scale Mf_π^2 has been removed [7,10]. In two-nucleon systems, \hat{d} plays a role analogous to that of g_A in the single-nucleon sector.

Model-independent correlations involving \hat{d} have been used to predict the rates of reactions which are important for the production and detection of solar neutrinos. In Ref. [7], the constant \hat{d} was fixed using the well-measured tritium beta-decay half-life. This allowed the authors of that paper to predict the rate of two reactions which are important members of the chain that produce energy in main-sequence stars and result in the emission of solar neutrinos: the proton-fusion process $pp \rightarrow de^+ \nu_e$ and the hep process ${}^3\text{He}p \rightarrow {}^4\text{He}e^+ \nu_e$. Importantly, \hat{d} also parametrizes the short-distance physics in the $\nu(\bar{\nu})d$ breakup reactions [11] that facilitate the flavor decomposition of the solar-neutrino signal from data obtained at the Sudbury Neutrino Observatory [12]. Since the pp fusion, hep, and $\nu(\bar{\nu})d$ breakup reactions occur at too low a rate to be reliably measured in the laboratory, the correlations

exhibited in Refs. [7,11] provide important, model-independent information on them. (This connection has also been explored in potential model calculations [13] and pionless EFT [14].) The same Gamow-Teller ${}^3S_1 \rightarrow {}^1S_0$ transition occurs in $\mu^- d \rightarrow nn\nu_\mu$, and the LEC \hat{d} plays a role there, too [15].

If we now insert the expression (5) into the Lagrangian (6), we see that if a pion is produced via the reaction $NN \rightarrow NN\pi$ and is in a p -wave relative to the NN system, then the LEC \hat{d} parametrizes the short-distance physics of the process. This is the two-body analog of the Goldberger-Treiman relation (1), and it establishes an intriguing connection between the reactions involved in the production and detection of solar neutrinos and the physics of pion production in NN collisions.

In systems with $A > 2$, \hat{d} plays a subtle, but important, role in nuclear binding through its generation of a three-nucleon force (3NF). As pointed out in Ref. [16], a 3NF is obtained if a virtual pion is produced via (6) and then absorbed on a third nucleon via the leading Lagrangian (4). The resulting diagram is a leading nonvanishing term in the chiral expansion of the 3NF [16,17] and is depicted in Fig. 1. Determining \hat{d} accurately from 3N data may be difficult [17], but the correlations mandated by the chiral symmetry of QCD mean that we can use the values of \hat{d} extracted from electroweak processes—e.g., in Ref. [7]—as input to fix the size of the 3NF depicted in Fig. 1. This has the advantage that the kinematics at which the virtual pion is emitted in Fig. 1 (pion energy ≈ 0) matches the kinematics at which reactions such as $pp \rightarrow de^+ \nu_e$ occur quite well—certainly much better than it does the kinematics of $NN \rightarrow NN\pi$, which was used to constrain the 3NF in Ref. [8]. This should reduce the impact that higher-order χ PT corrections have on predictions for this piece of the 3NF. Qualitative comparison of the values extracted for \hat{d} in Refs. [7,17] is encouraging, but for a quantitative comparison tritium β -decay and three-nucleon bound state and scattering calculations will have to be done with consistent wave functions and regulators.

Here we are less ambitious. We will instead exploit another correlation that results from substitution of (5) into (6)—one involving photons. We examine the reaction $\pi^- d \rightarrow nn\gamma$, which has played a significant role in determinations of the nn scattering length (a_{nn}). Data from radiative pion capture experiments dominate the accepted value $a_{nn} = -18.59 \pm 0.40$ fm [18–22]. Final-state interactions in nd breakup experiments can also be used to measure a_{nn} . While one such recent determination is in

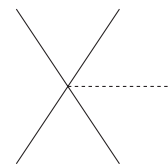


FIG. 1. The piece of the leading chiral 3NF which involves \hat{d} .

agreement with the accepted value, another measurement of a_{nn} using the same reaction disagrees with it by more than 4σ [23]. We will not comment on this further here but will focus on showing that there is a correlation between $\pi^- d \rightarrow nn\gamma$ and pp fusion. This correlation, together with knowledge of the pp fusion rate, reduces the impact of short-distance physics on the value of a_{nn} extracted from $\pi^- d \rightarrow nn\gamma$ data. This is important, because previous extractions of a_{nn} from this process [18,22] have shown that ± 0.3 fm out of the total uncertainty of ± 0.4 fm in the final result comes from the short-distance piece of the NN wave function. Up until now, it was not clear how to remove this source of uncertainty and so reduce the a_{nn} error below ± 0.3 fm.

We use wave functions that are derived in a manner consistent with chiral symmetry. For the deuteron S state, we start from the asymptotic wave function $u^{(0)}(r) = A_S e^{-\gamma r}$, where $A_S = 0.8846 \text{ fm}^{-1/2}$ is the asymptotic normalization and $\gamma = \sqrt{MB} = 45.7 \text{ MeV}$ the binding momentum [24]. For low-energy nn scattering, the starting point is asymptotic wave functions with phase shifts derived from the effective-range expansion, i.e., $v^{(0)}(r) = \sin(pr + \delta)/\sin\delta$, where p is the relative nn c.m. momentum and $p \cot\delta = -1/a_{nn} + \frac{1}{2}r_0 p^2$ (r_0 is the effective range). In both cases, the wave functions are calculated from $r = \infty$ down to a matching point R using the one-pion-exchange potential. For $r < R$, we then assume a spherical well potential and match to the $r > R$ wave function at $r = R$. Further details can be found in Refs. [22,25,26]. The pp state needed for the calculation of $pp \rightarrow de^+ \nu_e$ is calculated similarly, with the asymptotic wave function given by zero-energy Coulomb wave functions and the pp scattering length with respect to the Coulomb potential $a_{pp}^C = -7.8196 \pm 0.0026 \text{ fm}$ [27] as input. Higher-order electromagnetic corrections can be included and give a $\approx 1\%$ effect [13], but for our present purposes they are not important, since they alter only the long-distance properties of the pp wave function.

The matrix element for proton fusion is given by [28]:

$$\mathcal{M}_{\text{GT},1} = \frac{1}{A_S} \int dr u(r) v^C(r), \quad (7)$$

up to corrections of relative order χ^2 . In Eq. (7), $u(r)$ and $v^C(r)$ are the deuteron S -state and pp 1S_0 zero-energy scattering (including Coulomb) wave functions. The matrix element for the KR contribution to $\pi^- d \rightarrow nn\gamma$ in the final-state interaction (FSI) region (see Fig. 3) is

$$\mathcal{M}_{\text{FSI}} = C \int dr u(r) j_0\left(\frac{kr}{2}\right) v(r), \quad (8)$$

where C is a (known) constant, k the photon c.m. momentum, j_0 a spherical Bessel function, and $v(r)$ the full (energy-dependent) nn 1S_0 scattering wave function in the notation of Ref. [22]. Equation (8) is also of accuracy χ^2 .

Figure 2 then shows that the Gamow-Teller matrix element (7) is correlated with the FSI peak height in $\pi^- d \rightarrow$

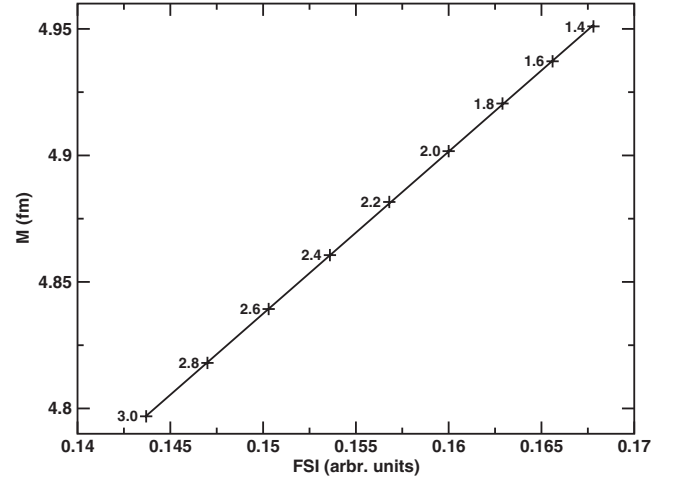


FIG. 2. Gamow-Teller matrix element (7) plotted against $\pi^- d \rightarrow nn\gamma$ FSI peak height given by (8) for various values of R . The nine points correspond to the values of R (in fm) indicated. The straight line is a linear fit to the points.

$nn\gamma$, which here, and in Fig. 3 below, we have computed using the $\mathcal{O}(\chi^3)$ χ PT amplitude as in Ref. [22]. The linear correlation in Fig. 2 suggests that an essentially R -independent result for the height of the FSI peak will be found if the LEC \hat{d} is adjusted so as to realize a particular value of the Gamow-Teller matrix element in pp fusion.

This is implemented in χ PT by adding to the matrix element (7) the contribution from \hat{d} :

$$\mathcal{M}_{\text{GT}}^{\text{LEC}} = \frac{1}{A_S} \hat{d} A_{3S1} \kappa_{3S1} A_{1S0}^C \kappa_{1S0}^C. \quad (9)$$

Here the short-distance ($r < R$) pieces of the deuteron and pp radial wave functions are given by $A_{3S1} \sin(\kappa_{3S1} r)$ and

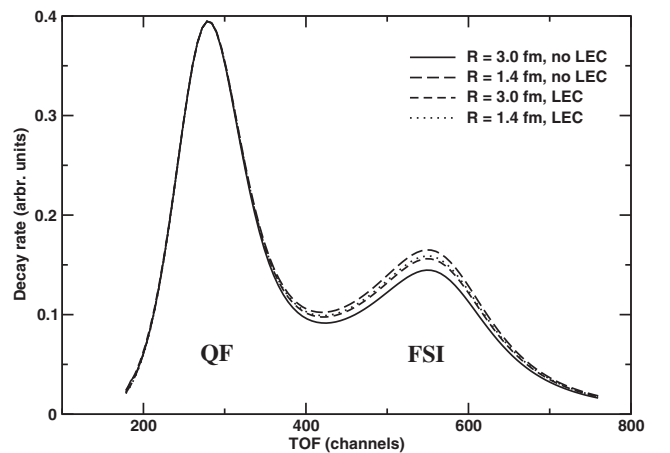


FIG. 3. Calculated $\pi^- d \rightarrow nn\gamma$ neutron time-of-flight distribution, for two R values, without and with the LEC \hat{d} contribution. The supplement of the γn laboratory angle is $\theta_3 = 0.075$ rad and $a_{nn} = -18.5$ fm. The labels QF (quasifree) and FSI indicate where the corresponding kinematics are dominant.

TABLE I. Values of \hat{d} for various matching radii R . Here we have used $f_\pi = 93.0$ MeV and $M = 939$ MeV.

R (fm)	1.4	1.8	2.2	2.6	3.0
\hat{d}	-1.27	-0.559	0.481	2.07	4.29

$A_{150}^C \sin(\kappa_{150}^C r)$. We then extract a value for \hat{d} by demanding that the total \mathcal{M}_{GT} agrees with the result $\mathcal{M}_{\text{GT}} = 4.898$ fm obtained from the Argonne v_{18} potential [29] in Refs. [13,25]. (If we choose the \mathcal{M}_{GT} found using other NN potentials in Ref. [13], the change in a_{nn} is <0.01 fm.) This procedure gives the values for $\hat{d}(R)$ displayed in Table I. These values are “natural” [30].

Now \hat{d} contributes to the FSI matrix element through

$$\mathcal{M}_{\text{FSI}}^{\text{LEC}} = C \frac{1}{A_S} \hat{d} A_{3S1} \kappa_{3S1} A_{150} \kappa_{150}, \quad (10)$$

where $A_{150} \sin(\kappa_{150} r)$ is the nn (radial) wave function for $r < R$. The $\pi^- d \rightarrow nn\gamma$ spectrum (both without and with the \hat{d} contribution) for varying radii R is then plotted in Fig. 3 [31].

Including \hat{d} drastically reduces the dependence on the short-range physics that dominated the a_{nn} uncertainty in Refs. [18,22]. If the entire spectrum is fitted, our R -related a_{nn} uncertainty is reduced from ± 1.0 to ± 0.14 fm. Other, smaller, sources of theoretical error in the a_{nn} extraction are the role of P waves in the nn final state, potential changes in wave functions when chiral two-pion exchange is incorporated, and pion-range $\mathcal{O}(Q^4)$ pieces of the two-body operator used to compute the spectrum of Fig. 3 [10,22]. Now that it is clear how to remove the R -related uncertainty, we will include these effects in our spectrum computation [25]. It should then be possible to extract a_{nn} from $\pi^- d \rightarrow nn\gamma$ with a total theoretical uncertainty of ≈ 0.3 fm if the entire spectrum is fitted and ≈ 0.05 fm if only the FSI peak is fitted. In the second case, the theoretical error would be more than 3 times smaller than in any previous extraction [18,19,22]. Experimental uncertainties would be the dominant contribution to the a_{nn} error bar. Note that the values found for a_{nn} by fitting in both regions should be consistent within errors—a useful test of the theory.

This demonstrates how χ PT links apparently disparate reactions in few-nucleon systems. The correlations that result from this are important not just for their impact on the accuracy with which the nn scattering length is known. They also mean that measurements of, e.g., $\gamma d \rightarrow nn\pi^+$ [32] might help determine the strength of the chiral 3NF and, if done with sufficient precision, could test the value of \hat{d} extracted from tritium beta decay in Ref. [7].

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