Entanglement of Two Impurities through Electron Scattering

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We study how two magnetic impurities embedded in a solid can be entangled by an injected electron scattering between them and by subsequent measurement of the electron's state. We start by investigating an ideal case where only the electronic spin interacts successively through the same unitary operation with the spins of the two impurities. We find conditions for the impurity spins to be maximally entangled with a significant success probability. We then consider a more realistic description which includes both the forward and backscattering amplitudes. In this scenario, we obtain the entanglement between the impurities as a function of the interaction strength of the electron-impurity coupling. We find that our scheme allows us to entangle the impurities maximally with a significant probability.

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Recently there has been an increasing interest in proposals for the generation of entanglement among spins in mesoscopic solid state structures [1–8]. The most natural schemes are for entangling *adjacent* stationary spins through a direct quantum gate [1,2]. A series of proposals in which one obtains a reasonable separation between the entangled spins have been proposed for *mobile* spins [3–8]. Proposals also exist for entangling orbital (first proposed in Ref. [5] and further explored in Refs. [6–8]) and path [9] degrees of freedom of mobile entities. However, in recent years it has become vital to envisage methods to entangle *stationary spins* (and stationary qubits in general) separated by a *distance longer than the range of their direct interaction*. The importance arises from the need to scale the power of quantum computers by linking distinct quantum registers [10,11] (if stationary qubits belonging to two distinct quantum registers are entangled, then the two registers can be effectively thought of as parts of the same quantum computer). Shuttling of ions or spins over a distance combined with precisely timed gates between shuttled and stationary ions or spins have been proposed for this purpose [10,11]. These operations require a high degree of control of interaction times. An important and challenging question which thus arises is whether it is possible to entangle stationary spins outside the range of each other's interactions under situations of *lower control*. One such example considered so far involves already starting with two entangled pairs of electrons [12]. Other examples are the cases in which a mobile spin spatially scatters when it interacts with stationary spins or when one cannot make a mobile spin interact differently with different stationary spins (say, as a result of its constant velocity while passing the stationary spins).

So far, most proposals of verifying entanglement in mesoscopic structures involve mobile entities in an essential way [5–9,13–15]. It would thus be of fundamental interest to create an entanglement in a solid which can be verified only by measurements on individual stationary spins [16].

With the above motivations is mind, in this Letter, we propose a scheme to entangle two magnetic impurities (stationary spins $1/2$) embedded in a solid state system. The main idea is to use a ballistic electron as an agent which scatters off the two impurities in succession and entangles them. Being a scattering based scheme, it requires no control over the ability to switch interactions on and off between entities in a solid, as is required by many existing entangling proposals [3]. Moreover, even in comparison to other reduced control proposals, such as those based on scattering or two particle interference [4], our current scheme has the simplicity that it involves only one mobile entity, namely, the ballistic electron, and does away with the difficulty of having to make two electrons coincide at the same place at the same time.

We comment first on the geometry of the system. Since entanglement generation depends on a conduction electron interacting with both impurities, it is most convenient to make the system's cross section as small as possible. In this spirit, and for the sake of simplicity, we consider a one-dimensional metallic atomic chain (of nonmagnetic

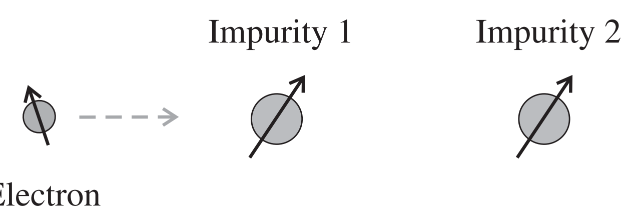

FIG. 1. Setup for our scheme to entangle two magnetic impurities of spin $1/2$ in a solid through electron scattering. The electron flies along a one-dimensional chain and interacts magnetically its with the impurities to entangle them. In a realistic scenario the electron may also spatially scatter from the impurities.

atoms), with two embedded (substitutional) spin-1/2 magnetic impurities. This is shown in Fig. 1.

We know that in an ideal case, where a mediating agent is allowed to interact with two systems through distinct unitary operations, it can then perfectly entangle them. The first of these unitaries perfectly entangles the first system with the agent, and then the second operation swaps the state of the agent with that of the second system. The different unitaries are implemented by different interaction times or strengths between the agent and each of the systems. Such a technique obviously requires either a great control over the motion of the agent, or the nontrivial engineering of different interaction strengths of the agent with the systems. Under these circumstances, it becomes interesting to investigate the reduced control situation where an agent interacts with both systems through the same unitary operation. How well can the systems be entangled under these circumstances? We first consider this simplified case, just in order to investigate how much entanglement can be established between two impurities, even when the electron interacts with them through the same unitary. We find that depending on whether the initial impurities are aligned or antialigned, one can obtain a highly or maximally entangled state with a significant probability. This case may not be realistic from the solid state scattering scenario, but it is an interesting precursor to the case when spatial scattering is involved. Moreover, a possible solid state scenario for this ideal case would be one in which an electron is carried by a surface acoustic wave (proposed recently for quantum computation [17]) of constant velocity and interacts magnetically with two identical impurities without any spatial scattering (just by virtue of passing close to the stationary spins). We then proceed to the realistic case of the electron being spatially scattered by the interaction with the impurities. Interestingly, in this case, we find that the electron can entangle the two impurities nearly perfectly (conditional on a favorable outcome of a measurement of the electron's spin). Moreover, the probability of this favorable outcome is significant (above 40%).

We begin by considering the ideal scenario where the electron's spin interacts in succession with each of the impurity spins through the Hamiltonian $H = J\vec{S} \cdot \vec{\sigma}$, where $\vec{\sigma}$ refers to the Pauli operators of the electronic spin, $\vec{S}$ refers to Pauli operators for the impurity spins, and J is the coupling constant between the spins. We now assume that the electron interacts with the two impurities in succession for equal intervals of time, so that with both impurities the same unitary operation is implemented. Let the joint unitary operation between the electron and impurity ι (with $\iota = 1, 2$) as a function of the interaction time t be denoted by $U_{e\iota}(t)$. We will first consider the initial state $|\psi_0\rangle = |\uparrow\rangle_e|\downarrow\rangle_1|\downarrow\rangle_2$ of the electron and the two impurities where the spin of the electrons are aligned with each other and antialigned with that of the impurity. The final state is then given by

$$\begin{aligned} |\psi_f\rangle &= U_{e2}(t)U_{e1}(t)|\psi_0\rangle \\ &= \frac{e^{-2iJt}}{2}(\alpha|\uparrow\rangle_e|\downarrow\rangle_1|\downarrow\rangle_2 + \beta|\downarrow\rangle_e|\uparrow\rangle_1|\downarrow\rangle_2 + \gamma|\downarrow\rangle_e|\downarrow\rangle_1|\uparrow\rangle_2), \end{aligned} \tag{1}$$

where $\alpha = (1 + e^{i4Jt})^2/2$, $\beta = (1 - e^{i8Jt})/2$, and $\gamma = 1 - e^{i4Jt}$. Note that if we now measure the spin of the electron and observe the state $|\downarrow\rangle_e$, the impurities will be left in the entangled state $\beta|\uparrow\rangle_1|\downarrow\rangle_2 + \gamma|\downarrow\rangle_1|\uparrow\rangle_2$. In Fig. 2 we present the probability of this outcome (dashed line), as well as the resulting amount of entanglement quantified by the entanglement of formation [18] (solid line) between the two impurities, both as a function of Jt, the product of the interaction strength, and the interaction time. We study the probability and the entanglement as a function of Jt in the interval $[0, \pi/2]$ as they are periodic functions, and observe that in this ideal model, maximal entanglement can only be generated with zero probability (as β and γ are not exactly equal in magnitude for any value of Jt). This, however, does not rule out the possibility of obtaining a high amount of entanglement with a significant probability, for example, an entanglement of 0.99 with a probability of 0.41 as seen from Fig. 2.

Despite the above, we would really like to generate maximal entanglement between the impurities. We thus consider the initial state $|\psi_0'\rangle = |\downarrow\rangle_e|\uparrow\rangle_1|\downarrow\rangle_2$ in which the impurities are antialigned. After the interaction of the electron with the two impurities via the same unitary, we obtain a state of the same form as Eq. (1) with coefficients $\alpha = \sin 4Jte^{i2Jt}$, $\beta = \cos 2Jt$, and $\gamma = 2\sin^2 2Jte^{i2Jt}$. Here β and γ can be equal in magnitude for certain values of Jt, and it is possible to project the impurities to a maximally

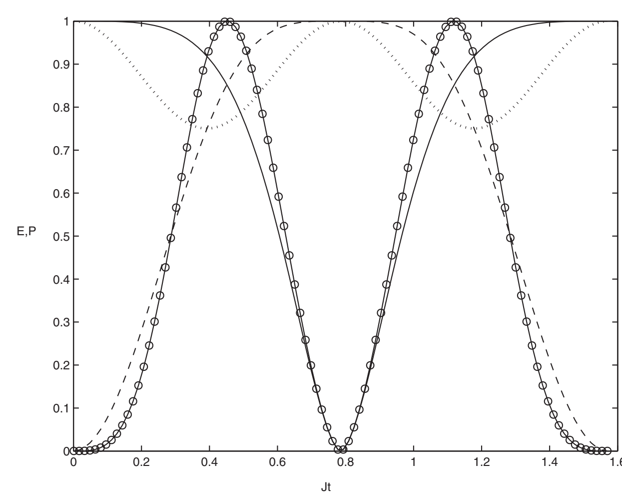

FIG. 2. Plots of the probability of success P (dashed line for aligned and dotted line for antialigned impurity spins) and entanglement E (solid line for aligned and line with circles for antialigned impurity spins) obtained between the impurity spins subject to success (detection of the electron spin to be down) in the ideal case of the electron successively interacting with the two impurities through the same unitary operation as a function of Jt.

entangled state corresponding to the outcome $|\downarrow\rangle_e$ of the electron's state. We have computed the probability of this outcome and the entanglement of formation of the state of the impurities thus generated and plotted these in Fig. 2 (probability of success as a dotted line and entanglement produced as a line with circles). We find that with a probability of 0.76, a maximally entangled state (entanglement = 1) can be generated.

Although the above protocols are of significantly reduced control in comparison to cases where time t_1 and t_2 can be made controllably different (such as by controlling the velocity of the electron), we have found that conditional generation of a maximally or near maximally entangled state of the two impurities is possible. However, one may ask how robust the protocol is with respect to *small* variations in t_1 and t_2 such that $t_2 = t_1 \pm \delta$. We have found that for both the above initial conditions, even for $\delta = 0.1t$, the overlap of the resulting state with the expected state (for $t_1 = t_2 = t$) is 0.998 at the values of Jt where maximally or near maximally entangled states are generated. Note that a quantum optical analogue (with cavities and flying atoms) of our initial condition $|\psi_0\rangle$ has been considered before with a slightly different (Jaynes-Cummings) Hamiltonian, where similar results (including the robustness to $t_2 = t_1 \pm \delta$) have been obtained [19]. One might thus expect that an analogue of initial condition $|\psi_0'\rangle$ might be able to generate a maximally entangled state in the same quantum optical system.

Let us now move to a more realistic scattering scenario. Magnetic impurities embedded in a conduction electron sea are traditionally modeled by an *s-d* Hamiltonian [20]. In this model the magnetic impurities are localized spins interacting with the conduction electrons via an exchange term. The full Hamiltonian of a system with one impurity reads

$$H = \sum_{k,\sigma} \varepsilon_k a_{k\sigma}^\dagger a_{k\sigma} + \sum_{kk'} J_{kk'} \vec{S}.\vec{s}_{kk'}, \quad (2)$$

where $\vec{S}$ is the impurity spin operator, $a_{k\sigma}^\dagger$ creates an electron with wave vector k and spin σ, and $\vec{s}_{kk'} = \hat{a}^\dagger \vec{\sigma}\, \hat{a}$, with

$$\hat{a} = \begin{pmatrix} a_{k\uparrow} \\ a_{k\downarrow} \end{pmatrix}.$$

The *s-d* Hamiltonian is actually derived from the more fundamental Anderson Hamiltonian through the Schrieffer-Wolff transformation. As a consequence, the interaction strength J is related to the strength of the Coulomb interaction between electrons and the hybridization of narrow and conduction bands [20]. In our calculation we will adopt the usual assumption that J is independent of k, k'.

We want to find out how much entanglement may be generated by a conduction electron that is injected in the system and interacts with both magnetic impurities. One may determine the system's final state by calculating the scattering matrix associated with each impurity and combining them together. The result is a sequence of (infinitely many) scattering processes, in which the output of a scattering event is the input of the subsequent one. The result of each individual scattering process is determined by use of Fermi's golden rule. The relevant T matrix is calculated to first order in the interaction.

If we consider that the conduction electron is being injected under low bias, its energy and wave vector will be the Fermi energy and Fermi wave vector of the system, respectively. We thus assume an initial state of the form

$$|\Psi_{\text{in}}\rangle = |k_F, \uparrow\rangle \otimes |\downarrow\downarrow\rangle, \quad (3)$$

which represents a conduction electron with positive Fermi wave vector k_F and spin $\uparrow$ propagating towards the two impurities, whose spins are both $\downarrow$.

As a result of the multiple scatterings of the conduction electron by the two impurities, a final state is generated which is a superposition of states in which the conduction electron has been reflected (r) or transmitted (t), $|\Psi_{\text{out}}\rangle = |\Psi_{\text{out}}^r\rangle + |\Psi_{\text{out}}^t\rangle$, and the transmitted component reads

$$|\Psi_{\text{out}}^t\rangle = A|k_F, \uparrow\rangle \otimes |\downarrow\downarrow\rangle + B|k_F, \downarrow\rangle \otimes |\uparrow\downarrow\rangle + C|k_F, \downarrow\rangle \otimes |\downarrow\uparrow\rangle. \quad (4)$$

The coefficients A, B, and C may be expressed as an infinite sum of powers of the product $J\rho(\varepsilon_F)$. For example, the coefficients up to sixth order in $J\rho(\varepsilon_F)$ (corresponding to three iterations of the scattering matrix) are $A^{(3)} = \frac{1}{N} \times (t^2 + t^2\lambda^2 - 8t\lambda^3 + 16\lambda^6 - 7t^2\lambda^4)$, $B^{(3)} = \frac{1}{N}(-2\lambda t + 2t\lambda^3 - 2t^2\lambda^2 + 6t\lambda^5 + 8t^2\lambda^4)$, and $C^{(3)} = \frac{1}{N}(-2\lambda t + 8\lambda^4 - 2t\lambda^3 + 2t^2\lambda^2 + 6t\lambda^5)$, where $\lambda = \pi i J\rho(\varepsilon_F)/2$, $t = 1 - \lambda$, and $\frac{1}{N} = \sqrt{|A^{(3)}|^2 + |B^{(3)}|^2 + |C^{(3)}|^2}$. We numeri-

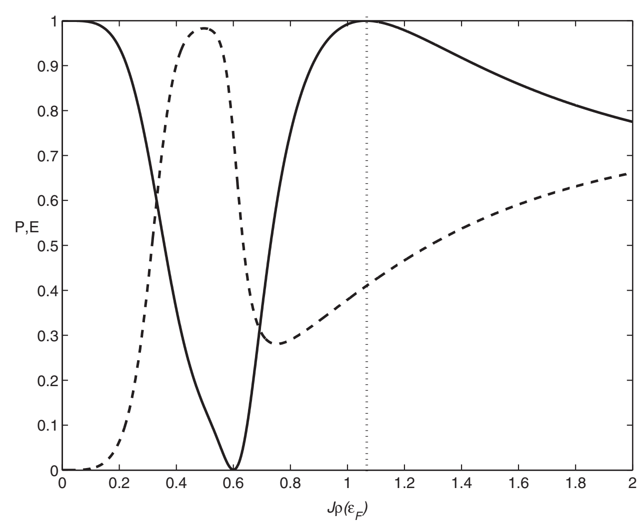

FIG. 3. Realistic case: Plots of the probability P (dashed line) of detecting the transmitted electron in the spin-down state, and the amount of entanglement E (solid line) obtained between the impurity spins in that case, as a function of the interaction strength $J\rho(\epsilon_F)$, for the realistic scenario of the electron successively scattering off the two impurities. Our results were calculated numerically to 14th order in the coupling constant, ensuring an accuracy of at least one part in 10^{-8}.

cally computed the series for $J\rho(\varepsilon_F) \in [0, 2]$ [according to our estimates [4], $J\rho(\varepsilon_F)$ is of the order of unit] and verified that the series converges rapidly in this domain. So essentially, the values of A, B, and C used in our calculations are correct for infinite iterations of the scattering matrix.

We now proceed to calculate the amount of entanglement generated *conditional on an electron being transmitted*, which is the entanglement contained in the state $|\Psi^t_{\mathrm{out}}\rangle$. Notice that if the transmitted electron has spin-up, the final state has zero entanglement. Thus we will evaluate only the entanglement of the state in which the transmitted electron has spin-down. Figure 3 shows the entanglement in this state (solid line) and the probability P of observing a transmitted electron with spin-down (dashed line). One may notice that there is some entanglement for most of the range $0 < J\rho(\varepsilon_F) < 2$, and the probability P is also considerable. Moreover, there are values of $J\rho(\varepsilon_F)$ for which the entanglement is maximum, and P is significant (0.41). Note that for this value of $J\rho(\varepsilon_F)$, even if we did not measure the spin of the transmitted electron, the state of the two impurities are in an entangled state. This entanglement can be verified by determining the expectation value of the witness operator $0.25[I \otimes I - 0.8212(\sigma_z \otimes I + I \otimes \sigma_z) - 0.5706(\sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y)]$ by local measurements [21] on the stationary impurity spins alone and finding to be negative.

In this Letter, we have presented a scheme for entangling two magnetic impurities (even maximally) in a solid through the scattering of a single ballistic electron and the subsequent detection of its spin. While much work has been done on entangling spins in mesoscopic solid state systems, this is the first proposal for entangling stationary spins which are well separated (i.e., outside the range of each other's direct interaction) using a reduced control method. Even if we did not measure the electron's spin and only did local measurements (of a witness operator) on the stationary impurity spins alone, then their entanglement could be verified, as opposed to the existing proposals for verifying entanglement between spins in mesoscopic structures [5–9,13–15]. A more significant consequence will be in interfacing different quantum registers for scaling quantum computers. The distance involved should be same as any other scheme involving ballistic mobile electrons [3,4]. The scheme should be implementable using the same systems as those used to study Kondo physics [22], and that the values of $J\rho(\varepsilon_F)$ needed for our scheme are achievable in these systems have been shown in Ref. [4].

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