Synchronization in the BCS Pairing Dynamics as a Critical Phenomenon

R. A. Barankov[1] and L. S. Levitov[2]

[1]Department of Physics, University of Illinois at Urbana-Champaign, 1110 W. Green St, Urbana, Illinois 61801, USA
[2]Department of Physics, Massachusetts Institute of Technology, 77 Massachusetts Ave, Cambridge, Massachusetts 02139, USA

(Received 12 March 2006; published 16 June 2006)

Fermi gas with time-dependent pairing interaction hosts several different dynamical states. Coupling between the collective BCS pairing mode and individual Cooper pair states can make the latter either synchronize or dephase. We describe transition from phase-locked undamped oscillations to Landau-damped dephased oscillations in the collisionless, dissipationless regime as a function of coupling strength. In the dephased regime, we find a second transition at which the long-time asymptotic pairing amplitude vanishes. Using a combination of numerical and analytical methods we establish a continuous (type II) character of both transitions.

DOI: 10.1103/PhysRevLett.96.230403 PACS numbers: 05.30.Fk, 03.75.Kk, 03.75.Lm, 03.75.Ss

Recent discovery of BCS pairing in fermionic vapors [1], made possible by control of interactions in trapped cold gases [2], has renewed interest in quantum collective phenomena [3]. Advanced detection techniques and long coherence times in vapors enable time-resolved studies of new collective modes, such as spin waves [4] and the BCS pairing mode [5].

Interaction between a collective mode and constituting particles is key for our understanding of dynamics in various systems, from plasma to quantum gases. One of the most surprising of these phenomena is Landau damping, which occurs in a collisionless regime via direct dissipationless energy transfer from the collective mode to single particles. Its nondissipative and thus reversible character [6] leads to a variety of regimes, notably to quenching of the damping, first explored in plasma physics [7]. Remarkably, a linearly damped mode can regrow and transform to a stationary oscillatory Bernstein-Greene-Kruskal mode. This fascinating prediction was confirmed experimentally only recently [8].

Naturally, the richness of these nonlinear phenomena makes it tempting to look for their analogs in cold gases. Collisionless damping in cold gases was considered, in the linear regime, for optical excitations [9], spin waves [10,11], and excitations in optical lattices [12]. Motivated by the work on fermion superfluidity [1,5], here we focus on the pairing dynamics of fermions [13–17] induced by a sudden change of interaction. The collisionless regime becomes practical in this case due to long relaxation times $\tau_\epsilon \gg \tau_\Delta = \hbar/\Delta$ [13], where Δ is the BCS gap, and $\tau_\epsilon \simeq \hbar E_F/\Delta^2$ is the two-fermion collision time estimated at the energy $\simeq \Delta$ near the Fermi level. The pairing mode of a small amplitude oscillates at a frequency $2\Delta/\hbar$ and exhibits its collisionless dephasing [18]. These conclusions were extended recently to the nonlinear regime [19].

This behavior changes drastically as the perturbation increases. The main result of this work, as summarized in Fig. 1, is prediction of a dynamical transition resulting from competition between synchronization and collisionless dephasing, taking place as a function of the initial pairing gap, Δ_s. We found three qualitatively different regimes (A, B, and C) with the critical points at $\Delta_{AB} = e^{-\pi/2}\Delta_0$ and $\Delta_{BC} = e^{\pi/2}\Delta_0$, where Δ_0 is the equilibrium pairing amplitude in the final BCS state. Below the A-B transition, $\Delta_s < \Delta_{AB}$, individual Cooper pair states synchronize and the pairing amplitude oscillates between Δ_+ and Δ_- without damping. In contrast, in the interval $\Delta_{AB} \le \Delta_s < \Delta_{BC}$ the pairing amplitude is Landau damped and exhibits decaying oscillation, saturating at an asymptotic value, Δ_a, with nonmonotonic dependence on Δ_s. A second transition occurs at $\Delta_s = \Delta_{BC}$. The dynamics becomes overdamped at $\Delta_s > \Delta_{BC}$, and $\Delta(t)$ decreases to zero without oscillations. The oscillation amplitude and the asymptotic value Δ_a vanish continuously at the critical points A-B and B-C, as in a type II transition. We demon-

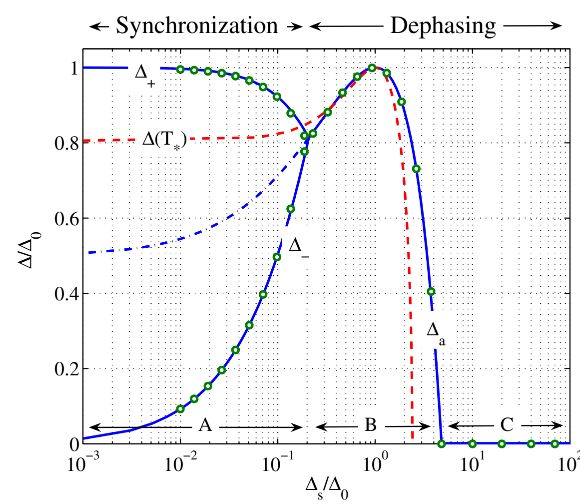

FIG. 1 (color online). Three regimes of the pairing dynamics vs the initial gap value Δ_s: numerical (open circles) and analytical (line). In synchronized phase (A), $\Delta_s < \Delta_{AB}$, the pairing amplitude oscillates between Δ_- and Δ_+. In the dephased regime (B, C), the pairing amplitude saturates to a constant value, Δ_a, when $\Delta_{AB} \le \Delta_s < \Delta_{BC}$, and decreases to zero at $\Delta_s \ge \Delta_{BC}$. Dashed line: The stationary gap value $\Delta(T_*)$ reached in a closed system after equilibration.

strate that these results are consistent with the spectral analysis [20] based on the integrability of the problem.

We also address the behavior on a long time scale, $t \gtrsim \tau_\epsilon$, after dissipation sets in. We find that energy relaxation in a closed system, such as an atom trap, makes it evolve to a new equilibrium state. Both the temperature T_* and the gap $\Delta(T_*)$ exhibit a nonmonotonic dependence on the initial conditions (Fig. 1).

In our analysis of the BCS problem we employ the well-known pseudospin formulation [21] in which spin 1/2 operators $s_{\mathbf{p}}^{\pm} = s_{\mathbf{p}}^x \pm i s_{\mathbf{p}}^y$ describe Cooper pairs $(\mathbf{p}, -\mathbf{p})$. The BCS Hamiltonian takes the form

$$\mathcal{H} = -\sum_{\mathbf{p}} 2\epsilon_{\mathbf{p}} s_{\mathbf{p}}^z - \lambda(t) \sum_{\mathbf{p},\mathbf{q}} s_{\mathbf{p}}^- s_{\mathbf{q}}^+, \quad (1)$$

where $\epsilon_{\mathbf{p}} = \mathbf{p}^2/2m - \mu$ is the free particle spectrum with μ the Fermi energy. Here we consider the time evolution induced by an instantaneous change of interaction from λ_s at $t < 0$ to λ at $t > 0$. In the spin formulation, Eq. (1), the dynamics is of a Bloch form

$$\frac{d\mathbf{r}_{\mathbf{p}}}{dt} = 2\mathbf{b}_{\mathbf{p}} \times \mathbf{r}_{\mathbf{p}}, \qquad \mathbf{b}_{\mathbf{p}} = -(\Delta_x, \Delta_y, \epsilon_{\mathbf{p}}), \quad (2)$$

where $\mathbf{r}_{\mathbf{p}} = 2\langle \mathbf{s}_{\mathbf{p}} \rangle$ are classical vectors, and the effective magnetic field $\mathbf{b}_{\mathbf{p}}$ depends on the pairing amplitude Δ. The latter is defined self-consistently:

$$\Delta = \Delta_x + i\Delta_y = \frac{\lambda(t)}{2} \sum_{\mathbf{p}} r_{\mathbf{p}}^+, \qquad r_{\mathbf{p}}^+ = r_{\mathbf{p}}^x + i r_{\mathbf{p}}^y. \quad (3)$$

We first present numerical results for the dynamics (2) and (3). The Runge-Kutta method of the 4th order was used with $N = 10^4, 10^5$ equally spaced discrete energy states within a band $W = 50\Delta_0$ with a constant density of states $\nu(E_F)$. As an initial state we take

$$r_{\mathbf{p}}^+(0) = \frac{\Delta_s}{\sqrt{\Delta_s^2 + \epsilon_{\mathbf{p}}^2}}, \qquad r_{\mathbf{p}}^z(0) = \frac{\epsilon_{\mathbf{p}}}{\sqrt{\Delta_s^2 + \epsilon_{\mathbf{p}}^2}}. \quad (4)$$

which describes the $T = 0$ paired ground state [21]. Without loss of generality we set $\Delta(t) = \Delta_x$, since the phase of Δ is a constant of motion due to the particle-hole symmetry of the model. The interaction constants λ_s and λ define, via the self-consistency relation (3), the initial and final equilibrium BCS gap values, $\Delta_s = We^{-1/g_s}$, $g_s = \lambda_s \nu \ll 1$, $\Delta_0 = We^{-1/g}$, $g = \lambda\nu$, which we use to parameterize the system.

We observe three qualitatively different dynamical regimes. The initial states with a relatively small gap give rise to undamped oscillations [Fig. 2(a)]. In this case $\Delta(t)$ oscillates nonharmonically between Δ_- and Δ_+ (the regime *A* in Fig. 1). Synchronization of different Cooper pair states results from their interaction with the mode singled out by BCS instability of the initial state, similar to the evolution from the normal state [13].

Desynchronization takes place at $\Delta_s \geq \Delta_{AB} = 0.21\Delta_0$ giving rise to two different regimes exhibiting dephasing, underdamped and overdamped (*B* and *C*, Fig. 1). The former, illustrated in Fig. 3(a), is simplest to understand

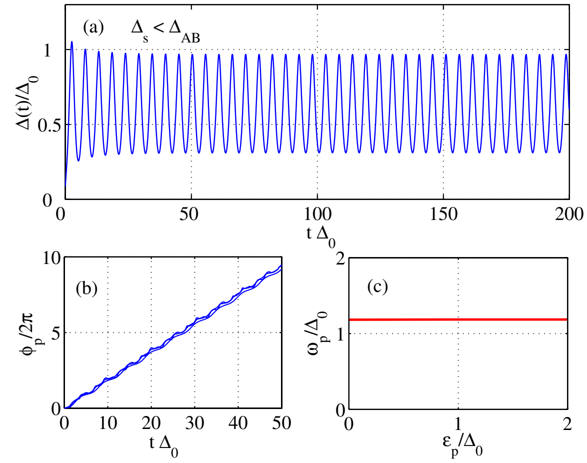

FIG. 2 (color online). (a) The pairing amplitude $\Delta(t)$ for the initial state (4) with $\Delta_s = 0.05\Delta_0$ as recorded from the simulation, oscillating between $\Delta_+ = 0.97\Delta_0$ and $\Delta_- = 0.31\Delta_0$; Synchronization (b) of the phase $\phi_{\mathbf{p}}$ time dependence, Eq. (7), for $\epsilon_{\mathbf{p}} = 0, \Delta_0, 2\Delta_0$, and (c) frequency $\omega_{\mathbf{p}}$ vs $\epsilon_{\mathbf{p}}$.

for a small initial deviation, $\Delta_s \simeq \Delta_0$ [18], by linearizing Bloch equations about the equilibrium state. The analysis predicts damped oscillations at long times:

$$\Delta(t) = \Delta_a + A(t)\sin(2\Delta_a t + \alpha), \qquad A(t) \propto t^{-1/2}. \quad (5)$$

The power-law decay of $A(t)$ was explained in Ref. [18] by interaction of the collective mode with the continuous spectrum of excitations with energies above $2\Delta_a$ and linked to the linear Landau damping. In the spin formulation, the dephasing results from the Larmor frequency of spin precession $\mathbf{b}_{\mathbf{p}}$ being a continuous function of $\epsilon_{\mathbf{p}}$. An extension of this argument to the nonlinear regime was proposed recently in Ref. [19] which, however, did not

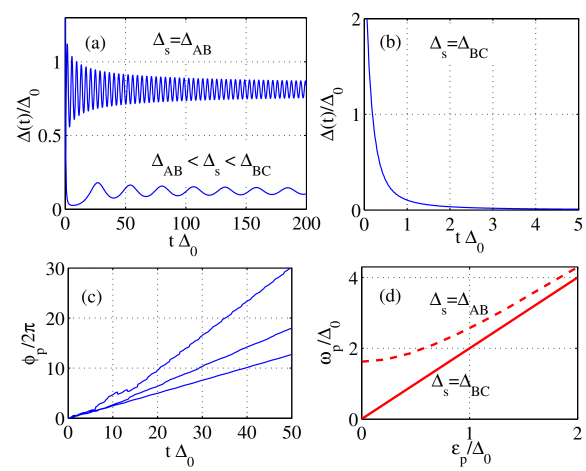

FIG. 3 (color online). Dephased dynamics. (a) Simulated $\Delta(t)$ for the initial states (4) with $\Delta_s = 0.21\Delta_0, 4.5\Delta_0$ with the asymptotic values $\Delta_a = 0.81\Delta_0, 0.12\Delta_0$; Overdamped dynamics. (b) Same as in (a) with $\Delta_s = 4.81\Delta_0$ and $\Delta_a = 0$; (c) The phase $\phi_{\mathbf{p}}$ time dependence, Eq. (7), for energies $\epsilon_{\mathbf{p}} = 0, \Delta_0, 2\Delta_0$ (bottom to top) for $\Delta_s = 0.21\Delta_0$; (d) The frequency $\omega_{\mathbf{p}}$ vs $\epsilon_{\mathbf{p}}$ for $\Delta_s = 0.21\Delta_0$ (dashed line) and $\Delta_s = 4.81\Delta_0$ (solid line).

clarify the range of its validity. The dephased time evolution similar to (5) was also reported in Refs. [15–17].

In the overdamped regime, $\Delta_s \geq \Delta_{BC} = 4.81\Delta_0$ [Fig. 3(b)], $\Delta(t)$ decays to zero without oscillations. This behavior can be understood in the limit $\Delta_s/\Delta_0 \gg 1$, i.e., when the coupling is turned off nearly completely. Treating different spins as precessing freely and independently, $r^+_{\mathbf{p}}(t) = e^{-i2\epsilon_{\mathbf{p}}t} r^+_{\mathbf{p}}(0)$, from Eq. (3) we find

$$\Delta(t \gg \Delta_s^{-1}) \propto (\Delta_s t)^{-1/2} e^{-2\Delta_s t}. \tag{6}$$

The fast dephasing can also be understood by noting that the energy distribution in (4) corresponds to an effective temperature $T \sim \Delta_s$ which exceeds T_c for Δ_0 (see below).

To fully exhibit phase locking in the synchronized regime which abruptly disappears in the dephased regime, we now explore the phase dynamics. It is convenient to measure precession angles relative to time-independent $\tilde{\mathbf{b}}_{\mathbf{p}} = -(\Delta_a, 0, \epsilon_{\mathbf{p}})$, where Δ_a is the asymptote $\Delta(t \to \infty)$ in the regimes B, C, and the average value of oscillating $\Delta(t)$ in A (dash-dotted line in Fig. 1). The angle and frequency of precession are defined by

$$n^+_{\mathbf{p}} = n^x_{\mathbf{p}} + i n^y_{\mathbf{p}} \propto e^{-i\phi_{\mathbf{p}}(t)}, \qquad \omega_{\mathbf{p}}(t) = d\phi_{\mathbf{p}}/dt, \tag{7}$$

where the vectors $n_{\mathbf{p}}$ are obtained from $r_{\mathbf{p}}$ by a rotation about the y axis which maps $\hat{z}$ onto $\tilde{\mathbf{b}}_{\mathbf{p}}$: $n^y_{\mathbf{p}} = r^y_{\mathbf{p}}$, $n^x_{\mathbf{p}} + i n^z_{\mathbf{p}} = e^{i\theta_{\mathbf{p}}}(r^x_{\mathbf{p}} + i r^z_{\mathbf{p}})$, with the rotation angle defined by $\tan\theta_{\mathbf{p}} = \Delta_a/|\epsilon_{\mathbf{p}}|$. The phase evolution, which becomes linear at long times $\tau \gg \Delta_0^{-1}$ (Figs. 2 and 3), can be characterized by average frequency (phase slope) $\omega_{\mathbf{p}} = \langle d\phi_{\mathbf{p}}/dt\rangle = [\phi_{\mathbf{p}}(\tau) - \phi_{\mathbf{p}}(0)]/\tau$. While in the regime A different $\mathbf{p}$ states phase lock [Figs. 2(b) and 2(c)], in the regimes B, C the frequencies $\omega_{\mathbf{p}}$ have dispersion [Figs. 3(c) and 3(d)]. The latter reproduces a quasiparticle spectrum, $\omega_{\mathbf{p}} = 2(\epsilon^2_{\mathbf{p}} + \Delta^2_a)^{1/2}$ with the long-time asymptote Δ_a which vanishes in the overdamped regime C.

We observe a qualitative change in behavior, with $\omega_{\mathbf{p}}$ dispersing in the regions B, C and phase locking in the region A. However, the oscillation amplitude $\frac{1}{2}(\Delta_+ - \Delta_-)$ in A and the asymptotic amplitude Δ_a in B vanish continuously at the critical points $\Delta = \Delta_{AB}, \Delta_{BC}$ (Fig. 1), indicating a type II transition.

Until now, we considered the dissipationless dynamics at times shorter than the quasiparticle relaxation time, $t \lesssim \tau_\epsilon$. Using the energy balance argument, one can determine the equilibrium state at a long time. To account for system equilibration, one needs to consider the full many-body Hamiltonian which enables elastic scattering of individual quasiparticles, omitted in Eq. (1). For a closed system, such as an atomic trap, the final temperature T_* and the gap Δ_* can be determined from the energy conservation condition $E_{t \gtrsim \tau_\epsilon}(T_*) = E_0$, where E_0 is the energy immediately after interaction switching,

$$E_0 = \sum_{\mathbf{p}} (\epsilon_{\mathbf{p}} - \tilde{\epsilon}_{\mathbf{p}}) + \frac{\Delta_s^2}{\lambda_s}\left(2 - \frac{\lambda}{\lambda_s}\right), \tag{8}$$

with the spectrum $\tilde{\epsilon}_{\mathbf{p}} = (\epsilon^2_{\mathbf{p}} + \Delta^2_s)^{1/2}$, and $E_{t \gtrsim \tau_\epsilon}(T_*)$ is the energy of the final state:

$$E_{t \gtrsim \tau_\epsilon}(T_*) = \sum_{\mathbf{p}} [\epsilon_{\mathbf{p}} - (1 - 2n_{\mathbf{p}})\tilde{\epsilon}_{\mathbf{p}}(\Delta_*)] + \frac{\Delta_*^2}{\lambda}. \tag{9}$$

Here $n_{\mathbf{p}} = 1/(1 + e^{\tilde{\epsilon}_{\mathbf{p}}(\Delta_*)/T_*})$ describes equilibrium with $T = T_*$, $\tilde{\epsilon}_{\mathbf{p}}(\Delta_*) = (\epsilon^2_{\mathbf{p}} + \Delta^2_*)^{1/2}$. After integrating over $\epsilon_{\mathbf{p}}$, we arrive at the equation for T_*:

$$F\left(\frac{\Delta_*}{2T_*}\right) = 1 - \left(\frac{\Delta_s}{\Delta_*}\right)^2 + \alpha\left(\frac{\Delta_s}{\Delta_*}\right)^2 \ln\left(\frac{\Delta_s}{\Delta_0}\right)^2, \tag{10}$$

where $F(u) = 2\int_0^\infty dx \cosh 2x[1 - \tanh(u\cosh x)]$, and $\alpha = 1 - g\ln(\Delta_s/\Delta_0)$. From Eq. (10) we obtain T_* and the equilibrium gap $\Delta_* = \Delta(T_*)$. Being approximately constant in the regime A, $T_* \approx 0.72 T_c$, $\Delta(T_*) \approx 0.81\Delta_0$, these quantities have nonmonotonic dependence in B, with Δ vanishing in C. The system turns normal in the final state for $\Delta_s \geq f(g)\Delta_0$, $f(g) = 2.1 + 0.8g + O(g^2)$. The numerical solution of Eq. (10) at $g = 0.26$ is displayed in Fig. 1 (dashed line).

It is instructive to compare our results to the spectral analysis based on the integrability of the BCS Hamiltonian [20,22,23]. There is an infinite number of commuting integrals of motion, $R_{\mathbf{p}} = \mathbf{L}_{\mathbf{p}}\mathbf{s}_{\mathbf{p}}$, parameterized by $\mathbf{p}$, where, following Ref. [20], we employ the Lax vector,

$$\mathbf{L}_{\mathbf{p}} = \hat{z} + \lambda \sum_{\mathbf{p}' \neq \mathbf{p}} \frac{\mathbf{s}_{\mathbf{p}'}}{\epsilon_{\mathbf{p}} - \epsilon_{\mathbf{p}'}}. \tag{11}$$

We will need $\mathbf{L}^2_{\mathbf{p}} = 4(\mathbf{L}_{\mathbf{p}}\mathbf{s}_{\mathbf{p}})^2$ which is also conserved [due to the Pauli matrices algebra combined with Eq. (11)]. The mean-field expressions are obtained by substituting the averages $\langle \mathbf{s}_{\mathbf{p}}\rangle$ instead of $\mathbf{s}_{\mathbf{p}}$.

The spectral polynomial defining the evolution of individual states is proportional to the square of the Lax vector [20], $Q(\epsilon) = \mathbf{L}^2(\epsilon)\prod_{\mathbf{p}}(\epsilon - \epsilon_{\mathbf{p}})^2$, where $\mathbf{L}(\epsilon_{\mathbf{p}}) = \mathbf{L}_{\mathbf{p}}$. The pairs of complex roots of the spectral equation $Q(\epsilon) = 0$ uniquely determine the long-time dynamics of the system [20]. Evaluation of $\mathbf{L}^2(y)$ for the initial state (4) gives

$$\mathbf{L}^2(y) = g^2\left(\ln\frac{\Delta_s}{\Delta_0} + yG(y)\right)^2 + g^2 G^2(y), \tag{12}$$

$$G(y) = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{y - \sinh x} = \frac{1}{2\sqrt{1 + y^2}} \ln\frac{y + \sqrt{1 + y^2}}{y - \sqrt{1 + y^2}},$$

where $y = \epsilon/\Delta_s$ is complex. One can see that the roots of $\mathbf{L}^2(y)$ lie on the imaginary axis. Upon variation of Δ_s they disappear at $y = 0$. Expanding about $y = 0$, we obtain

$$\mathbf{L}^2(y) = g^2\left(\ln\frac{\Delta_s}{\Delta_0} - \frac{\pi}{2}\right)\left(\ln\frac{\Delta_s}{\Delta_0} + \frac{\pi}{2}\right) + O(y). \tag{13}$$

Thus $\mathbf{L}^2(y)$ has no complex roots at $\Delta_s/\Delta_0 \geq e^{\pi/2}$, one pair of roots $y = \pm iu$ when $e^{-\pi/2} \leq \Delta_s/\Delta_0 < e^{\pi/2}$, with another pair appearing at $\Delta_s/\Delta_0 < e^{-\pi/2}$.

There is a direct correspondence between this behavior of the roots and the dynamical regimes A, B, and C

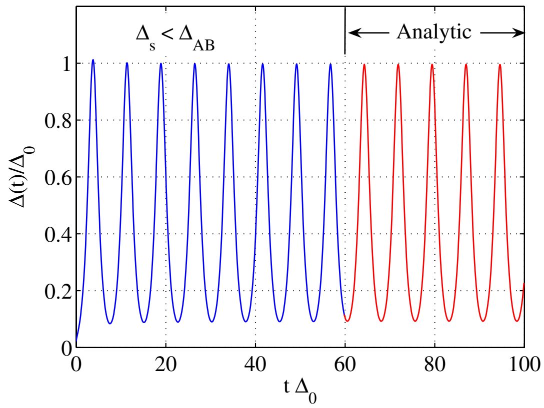

FIG. 4 (color online). The pairing amplitude $\Delta(t)$ as a function of time for the initial state (4) with $\Delta_s = 0.01\Delta_0$ as recorded from numerical simulation. At long time, numerical $\Delta(t)$ matches the analytic form in Eq. (14), oscillating nonharmonically between $\Delta_+ = 0.997\Delta_0$ and $\Delta_- = 0.093\Delta_0$.

observed numerically. The pairing amplitude is subject to fast dephasing and tends to zero when $\mathbf{L}^2(y)$ does not have complex roots. A pair of complex roots $y_a = \pm i\Delta_a/\Delta_s$ defines the long-time asymptote $\Delta(t) \approx \Delta_a$. Two pairs of roots, $y = \pm iu_1$ and $y = \pm iu_2$, correspond to the parameters $\Delta_\pm = (u_1 \pm u_2)\Delta_s$ of the Jacobi elliptic function [24] which defines the asymptotic behavior:

$$\Delta(t) = \Delta_+ \,\mathrm{dn}[\Delta_+(t - \tau_0), k], \qquad k = 1 - \Delta_-^2/\Delta_+^2, \quad (14)$$

where the time lag τ_0 is a half of the period. As illustrated in Fig. 4, Eq. (14) agrees well with $\Delta(t)$ found numerically. Thus the spectral analysis is in accord with the simulation of Bloch dynamics. It confirms the existence of the three regimes and also provides the exact values $\Delta_{AB} = e^{-\pi/2}\Delta_0$ and $\Delta_{BC} = e^{\pi/2}\Delta_0$.

Finally, we estimate the change of scattering length required to cross the *A*-*B* and *B*-*C* transitions. Using the BCS gap in a weakly interacting Fermi gas [25], $\Delta = 0.49E_F e^{-1/g}$, $g = \frac{2}{\pi}k_F|a|$, we see that the conditions $\Delta_s/\Delta_0 = e^{\pm\pi/2}$, written as $1/g - 1/g_s = \pm\pi/2$, translate into $1/a - 1/a_s = \pm k_F$. At weak coupling this corresponds to a small change of scattering length, $\delta a/a \approx \pm k_F a$, easily achievable for magnetically tunable Feshbach resonance. In such an experiment, at the times shorter than the thermalization time, one expects to observe the two regimes of dissipationless dynamics discussed above, oscillatory and dephased. The amplitude of the oscillations persistent at long times vanishes continuously at the threshold as in a type II transition.

We are grateful to Boris Spivak and Kumar Raman for helpful discussions.

Note added in proof.—Recently, the *B*-*C* crossover and the properties of the fully dephased regime *C* were explored independently by Yuzbashyan and Dzero [26].

[1] C. A. Regal, M. Greiner, and D. S. Jin, Phys. Rev. Lett. **92**, 040403 (2004); M. W. Zwierlein, C. A. Stan, C. H. Schunck, S. M. F. Raupach, A. J. Kerman, and W. Ketterle, Phys. Rev. Lett. **92**, 120403 (2004).
[2] E. Timmermans, P. Tommasini, M. Hussein, and A. Kerman, Phys. Rep. **315**, 199 (1999).
[3] F. Dalfovo, S. Giorgini, L. P. Pitaesvkii, and S. Stringari, Rev. Mod. Phys. **71**, 463 (1999).
[4] H. J. Lewandowski, D. M. Harber, D. L. Whitaker, and E. A. Cornell, Phys. Rev. Lett. **88**, 070403 (2002); J. M. McGuirk, H. J. Lewandowski, D. M. Harber, T. Nikuni, J. E. Williams, and E. A. Cornell, Phys. Rev. Lett. **89**, 090402 (2002).
[5] C. Chin, M. Bartenstein, A. Altmeyer, S. Riedl, S. Jochim, J. H. Denschlag, and R. Grimm, Science **305**, 1128 (2004).
[6] R. W. Gould, T. M. O'Neil, and J. H. Malmberg, Phys. Rev. Lett. **19**, 219 (1967).
[7] I. B. Bernstein, J. M. Green, and M. D. Kruskal, Phys. Rev. **108**, 546 (1957).
[8] J. R. Danielson, F. Anderegg, and C. F. Driscoll, Phys. Rev. Lett. **92**, 245003 (2004).
[9] M. Ö. Oktel and L. S. Levitov, Phys. Rev. Lett. **83**, 6 (1999).
[10] T. Nikuni and J. E. Williams, J. Low Temp. Phys. **133**, 323 (2003).
[11] R. J. Ragan, W. J. Mullin, and E. B. Wiita, cond-mat/0502189.
[12] S. Tsuchiya and A. Griffin, Phys. Rev. A **72**, 053621 (2005).
[13] R. A. Barankov, L. S. Levitov, and B. Z. Spivak, Phys. Rev. Lett. **93**, 160401 (2004).
[14] A. V. Andreev, V. Gurarie, and L. Radzihovsky, Phys. Rev. Lett. **93**, 130402 (2004); R. A. Barankov and L. S. Levitov, Phys. Rev. Lett. **93**, 130403 (2004).
[15] M. H. S. Amin, E. V. Bezuglyi, A. S. Kijko, and A. N. Omelyanchouk, Low Temp. Phys. **30**, 661 (2004).
[16] G. L. Warner and A. J. Leggett, Phys. Rev. B **71**, 134514 (2005).
[17] M. H. Szymańska, B. D. Simons, and K. Burnett, Phys. Rev. Lett. **94**, 170402 (2005).
[18] A. F. Volkov and Sh. M. Kogan, Zh. Eksp. Teor. Fiz. **65**, 2038 (1973) [Sov. Phys. JETP **38**, 1018 (1974)].
[19] E. A. Yuzbashyan, O. Tsyplyatyev, and B. L. Altshuler, Phys. Rev. Lett. **96**, 097005 (2006).
[20] E. A. Yuzbashyan, B. L. Altshuler, V. B. Kuznetsov, and V. Z. Enolskii, Phys. Rev. B **72**, 220503 (2005).
[21] P. W. Anderson, Phys. Rev. **112**, 1900 (1958).
[22] R. W. Richardson and N. Sherman, Nucl. Phys. **52**, 221 (1964).
[23] M. C. Cambiaggio, A. M. F. Rivas, and M. Saraceno, Nucl. Phys. A **624**, 157 (1997).
[24] *Handbook of Mathematical Functions*, edited by M. Abramowitz and I. A. Stegun (U.S. GPO, Washington, D.C., 1972).
[25] L. P. Gor'kov and T. K. Melik-Barkhudarov, Zh. Eksp. Teor. Fiz. **40**, 1452 (1961) [Sov. Phys. JETP **13**, 1018 (1961)].
[26] E. A. Yuzbashyan, and M. Dzero, Phys. Rev. Lett. **96**, 230404 (2006).