Biased Tomography Schemes: An Objective Approach

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We report on an intrinsic relationship between the maximum-likelihood quantum-state estimation and the representation of the signal. A quantum analogy of the transfer function determines the space where the reconstruction should be done without the need for any *ad hoc* truncations of the Hilbert space. An illustration of this method is provided by a simple yet practically important tomography of an optical signal registered by realistic binary detectors.

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The development of effective and robust methods of quantum-state reconstruction is a task of crucial importance for quantum optics and information. Such methods are needed for quantum diagnostics: for the verification of quantum-state preparation, for the analysis of quantum dynamics and decoherence, and for information retrieval. Since the original proposal for quantum tomography and its experimental verification [1,2] this discipline has recorded significant progress and is considered as a routine experimental technique nowadays. Reconstruction has been successfully applied to probing the structure of entangled states of light and ions, operations (quantum gates) with entangled states of light and ions or internal angular momentum structure of correlated beams, just to mention a few examples [3].

All these applications exhibit common features. Any successful quantum tomography scheme relies on three key ingredients: on the availability of a particular tomographically complete measurement, on a suitable representation of quantum states, and on an adequate mathematical algorithm for inverting the measured data. In addition, the entire reconstruction scheme must be robust with respect to noise. In real experiments the presence of noise is unavoidable due to losses and due to the fact that detectors are not ideal. The presence of losses poses a limit on the accuracy of a reconstruction. However, the very presence of losses can be turned into advantage and used for the reconstruction purposes. As has been predicted in Ref. [4], imperfect detectors, which are able to distinguish only between the presence and absence of signal (binary detectors) provide sufficient data for the reconstruction of the quantum state of a light mode provided their quantum efficiencies are less than 100%. The presence of losses is thus a necessary condition for a successful reconstruction: An ideal binary detector would measure only the probability of finding the signal in the vacuum state.

The required robustness of a tomography scheme with respect to noise is often difficult to meet especially if it is biased, that is, if some aspects of the quantum systems in question are observed more efficiently than the others. Since our ability to design and control measurements is severely limited, this situation will typically arise when one wants to characterize a system with a large number or infinitely many degrees of freedom, for instance in the quantum tomography of light mode mentioned above. The standard approach is to truncate the Hilbert space by a certain cutoff, reducing drastically the number of parameters involved [5]. Needless to say, such *ad hoc* truncation lacks physical foundation. It may have bad impacts on the accuracy of reconstruction or conversely it may lead to more regular results. The latter case may easily happen when an experimentalist seeks for the result in the neighborhood of the true state. Such a tacitly accepted assumption may appear as crucial as it allows elimination of the infinite number of unwanted free parameters. This drawback erodes the notion of tomography as an objective scheme.

In this Letter we propose a reconstruction procedure that is optimized with respect to the experimental setup, representation and inversion, designed for dealing with biased tomography schemes. The recommended approach to the generic problem of quantum-state tomography will be demonstrated on the scheme of a light mode adopting elements of linear optics (beam splitter) with realistic binary detectors detecting the presence or absence of the signal only. In addition, we for the first time present a statistically correct description of such a tomographic scheme.

Let us develop a generic formalism for the maximum-likelihood (ML) inversion of the measured data. Let us assume detections of a signal enumerated by the generic index j. Their probabilities are predicted by quantum theory by means of positive-operator-valued measure (POVM) elements $\mathbf{A}_j$,

$$p_j = \mathrm{Tr}[\mathbf{A}_j \rho], \qquad 0 \le \mathbf{A}_j \le 1, \tag{1}$$

ρ being the quantum state. The observations $\mathbf{A}_j$ are assumed to be tomographically complete in the Hilbert subspace we are interested in. No other specific assumptions about the operators $\mathbf{A}_j$, their commutation relations or group properties will be made. In general, probabilities p_j are not normalized to one as the operator sum

$$\sum_j \mathbf{A}_j = G \geq 0 \tag{2}$$

may differ from the identity operator. Theoretical probabilities p_j can be sampled experimentally by means of registered data N_j. The aim is to find the quantum state ρ from data N_j.

The ML scenario hinges upon a likelihood functional associated with the statistics of the experiment. In the following, we will adopt the generic form of likelihood for un-normalized probabilities [6]

$$\log \mathcal{L} = \sum_j N_j \log\left[\frac{p_j}{\sum_{j'} p_{j'}}\right], \tag{3}$$

which should be maximized with respect to ρ. Here the index j runs over all registered data. The extremal equation for the maximum-likely state can be derived in three steps: (i) The positivity of ρ is made explicit by decomposing it as $\rho = \sigma^\dagger \sigma$. (ii) Likelihood (3) is varied with respect to independent matrix σ using $\delta(\log p_j)/\delta\sigma = \mathbf{A}_j \sigma^\dagger / p_j$; (iii) Obtained variation is set equal to zero and multiplied from right side by σ with the result

$$R\rho = G\rho, \qquad R = \sum_j \frac{\sum_{j'} p_{j'}}{\sum_{j'} N_{j'}} \frac{N_j}{p_j(\rho)} \mathbf{A}_j, \tag{4}$$

where the operator G is defined by Eq. (2) and operator R depends on the particular choice of $\mathcal{L}$. Notice that this equation may be cast in the form of expectation-maximization (EM) algorithm [7]

$$R_G \rho_G = \rho_G, \tag{5}$$

where $R_G = G^{-1/2} R G^{-1/2}$ and $\rho_G = G^{1/2} \rho G^{1/2}$. This extremal equation may be solved by iterations in a fixed orthogonal basis. Keeping the positive semidefiniteness of ρ_G [by combining Eq. (4) with its Hermitian conjugate] the $(n+1)$th iteration reads

$$\rho_G^{(n+1)} = R_G^{(n)} \rho_G^{(n)} R_G^{(n)}, \qquad R_G^{(n)} = G^{-1/2} R(\rho^{(n)}) G^{-1/2}.$$

Starting with some initial guess $\rho_G^{(0)}$ the iterations are repeated until the fixed point is reached. In terms of ρ_G, the desired solution is then given by

$$\rho = G^{-1/2} \rho_G G^{-1/2}. \tag{6}$$

Going back to likelihood in Eq. (3) we now see that the operator G coming from the mutual normalization of probabilities, $\sum_j p_j = \mathrm{Tr}[\rho G]$, provides a complete (normalized) POVM, which is equivalent to the original biased observations A_j: $\sum_j G^{-1/2} A_j G^{-1/2} = 1_G$. This establishes the preferred basis for a reconstruction. Due to the division by the operator G in Eq. (6) and in the sentence above the reconstruction can be done only in the subspace spanned by the nonzero eigenvalues of G. The spectrum of G plays, therefore, the role of tomographic transfer function analogously to the transfer function in optical imaging. It quantifies the resolution of the reconstruction in the Hilbert space. A large eigenvalue of G indicates that many observations overlapped in the corresponding Hilbert subspace and this part of the Hilbert space is more visible. The Hilbert subspace where the reconstruction was done is clearly not a subject of a free choice in the proper statistical analysis. This is the main result of this Letter. This also gives a clue how to approximate the solution in the infinite dimensional case simply by taking the subspace corresponding to the dominant eigenvalues. The result of reconstruction can be easily checked in the preferred basis afterwards. If the reconstructed state exhibits dominant contributions for the components with relatively small eigenvalues of G, the result cannot be trusted.

The essence of the correct reconstruction inhere in the following recommended scenario: After collecting all data the optimal basis for reconstruction is identified as eigenvectors of G operator. The truncation is achieved by taking into account only those with dominant eigenvalues, where the ML extremal equation should be solved keeping the semipositive definiteness of the density matrix. This establishes the quantum tomography as an objective tool for the analysis of infinite dimensional quantum systems. Indeed, previously reported results of tomographic schemes have always considered the space for reconstruction *ad hoc*: If one knows what the result should be it is not really difficult to get it.

Let us illustrate this procedure on the following simple realistic detection setup: the signal state (described by the density matrix ρ) of the input mode a is mixed on a beam splitter with the probe coherent state $|\beta\rangle$ of the mode b and the mixed field is detected on a single on-off detector. Then the probability p of having *no* counts on the detector is measured.

Such nonideal measurements have already been used for tomography purposes. The inference of a photon number distribution was proposed in Ref. [8] and experimentally realized in Ref. [9]. A more advanced setup based on a multichannel fiber loop detector was developed and experimentally verified earlier in Ref. [10]. As proposed in Refs. [11,12], the reconstruction of a full density matrix can be done by measuring a coherently shifted signal. This reconstruction technique has also been implemented experimentally as a direct counting of Wigner function [13]. However, the algorithms used for the quantum-state reconstruction were not robust as indicated by the fact that they could give nonphysical results. This is due to the *a priori* constraints put on a quantum object, namely, the semipositive definiteness of a density matrix $\rho \geq 0$, which is not guaranteed in the above mentioned schemes. While it seems to be intractable to implement the condition of positive semidefiniteness in Wigner representation, it can be done in the general formalism adopting the maximum-likelihood estimation.

The probability of registering no counts on the detector is given by Mandel's formula [14]:

$$p = \langle :\ \exp\{-\nu_c c^\dagger c\}\ :\rangle, \tag{7}$$

where ν_c is the efficiency of the detector; $c^\dagger$ and c are creation and annihilation operators of the output mode, and :: denotes the normal ordering. For simplicity, we assume here that in the absence of the signal the detector does not produce any clicks; dark count are ignored. Let us assume that the beam-splitter transforms input modes a and b in the following way: $c = a\cos(\alpha) + b\sin(\alpha)$. Averaging over the probe mode b, from Eq. (7) one obtains

$$p = \sum_{n=0} (1-\bar{\nu})^n \langle n|D^\dagger(\gamma)\rho D(\gamma)|n\rangle, \tag{8}$$

where $\bar{\nu} = \nu_c \cos^2(\alpha)$, $\gamma = -\beta\tan(\alpha)$, $D(\gamma) = \exp\{\gamma a^\dagger - \gamma^* a\}$ is the coherent shift operator, and $|n\rangle$ denotes a Fock state of the signal mode a. Using the operator notation $\mathbf{R}_{n,\gamma} = D(\gamma)|n\rangle\langle n|D^\dagger(\gamma)$, the no-count probability is generated by the POVM elements $\mathbf{A}_{\nu,\gamma} = \sum_n (1-\nu)^n \mathbf{R}_{n,\gamma}$ and, defining a collective index $j = \{\nu, \gamma\}$, the counted probability coincides with Eq. (1).

Figure 1 shows how a suitable choice of γ points for a fixed truncation number $N_{\text{tr}} = 15$ can be achieved. Obviously, the amount of data used in Fig. 1(a) as compared to Fig. 1(b) is excessive for the reconstruction. On the other hand, when the number of points is too small, or they are chosen in an inappropriate way, eigenvalues of G differ strongly making reconstruction unfeasible. For example, in Fig. 1(d) the last eigenvalue is only $\sim 10^{-5}$. However, one needs to mention that the analysis of G provides a necessary but not sufficient condition of the reconstruction feasibility. In particular, a single γ point measurement is not sufficient (just like a measurement in $\gamma = 0$ is able to give only the diagonal elements). One needs to measure in at least two different nonzero γ points. The confidence interval on the reconstructed density matrix elements can be provided with help of variance $\sigma(\rho_{mn}) = [F(\rho_{mn})N_{\text{mes}}]^{-1/2}$, where N_{mes} is the total number of measurements, and the Fisher information F can be defined for real part of the density matrix elements as [15]

$$F(\text{Re}[\rho_{mn}]) = \sum_j \frac{\sum_{j'} p_{j'}}{p_j}\left[\frac{\partial}{\partial\,\text{Re}(\rho_{mn})}\frac{p_j}{\sum_{j'} p_{j'}}\right]^2, \tag{9}$$

and similarly with Re changed to Im for imaginary part of ρ.

To illustrate our discussion, let us consider a reconstruction of the following state (Fig. 2):

$$|\phi\rangle = (|0\rangle + \exp\{0.5i\}|2\rangle)/\sqrt{2}. \tag{10}$$

The simulation was done using a total of 10^7 measurements collected in five different points on the phase plane γ. In Fig. 2(a) one can see the eigenvalues of the matrix G (2). Obviously, the chosen set of points is suitable for the

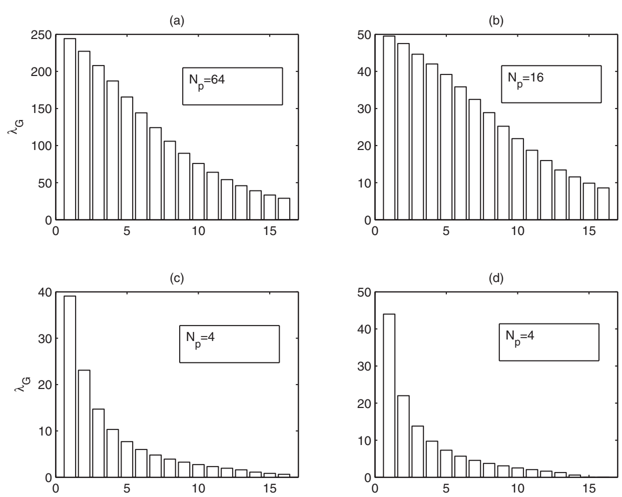

FIG. 1. Eigenvalues of the matrix G (2) truncated at $N_{\text{tr}} = 15$. The simulated measurement was done at N_p γ points equidistantly distributed in regions: (a) and (b) $\text{Re}(\gamma) \in [-2, 2]$, $\text{Im}(\gamma) \in [-2, 2]$; (c) $\text{Re}(\gamma) \in [-1, 1]$, $\text{Im}(\gamma) = 0$; (d) $\text{Re}(\gamma) \in [1, 1.01]$, $\text{Im}(\gamma) = 0$. In all panels, 10 equidistant values of the detector efficiency were chosen from the interval $\eta \in [0.1, 0.9]$.

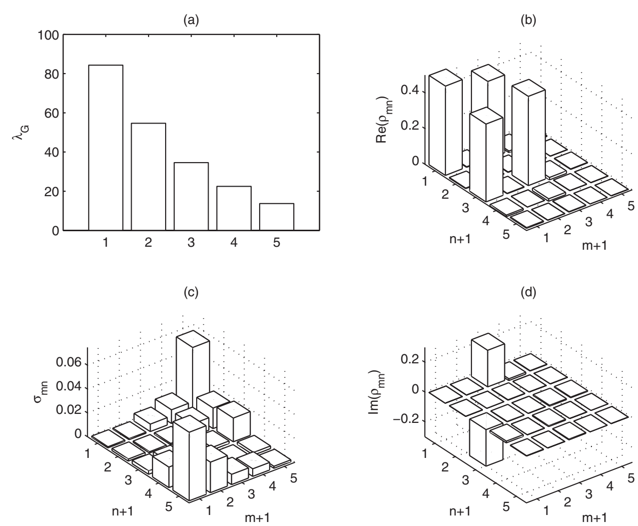

FIG. 2. A reconstruction of the state (10) according to procedure (5). The following measurements were used: $\text{Re}(\gamma) = -0.2, -0.1, 0, 0.1, 0.2$; $\text{Im}(\gamma) = 0.1, -0.5, 0, 0.5, 0.1$; 20 equidistantly distributed detector efficiencies in the interval $[0.1, 0.9]$ were used; the Hilbert space was truncated at $N_{\text{tr}} = 5$. Panel (a) shows the eigenvalues of the matrix G. Panels (b) and (d) show the real and imaginary parts of the reconstructed matrix (in Fock basis). They were obtained using 10^6 iteration of the EM algorithm. Panel (c) shows the variances of the real part ($n \leq m$) and imaginary part ($n > m$) of the reconstructed elements given by Eq. (9).

reconstruction. Notice the correlation between decreasing eigenvalues and increasing errors in Figs. 2(a) and 2(c).

This objective approach may be compared with alternative schemes based on the reconstruction of Wigner function. Measurement in any given γ point is able to give a value of the Wigner function in that point. Indeed [16],

$$W(\gamma) = \frac{2}{\pi}\sum_{n=0}(-1)^n R_{n,\gamma}, \tag{11}$$

where $R_n(\gamma) = \mathrm{Tr}[\rho \mathbf{R}_{n,\gamma}] \equiv \langle n|D^\dagger(\gamma)\rho D(\gamma)|n\rangle$. For a fixed value of the amplitude γ one should seek the set of non-negative matrix elements $R_{n,\gamma}$ and plug in these values into the definition of the Wigner function (11). These matrix elements can be found by inverting the counted statistics (8) measured with a set of different efficiencies solving a linear positive inverse problem. This can be accomplished by means of the EM algorithm similarly to the approach used in Ref. [17]. An example of such a reconstruction is shown in Fig. 3. Though the reconstruction seems to be faithful, one should keep in mind that even very small deviations from the true Wigner function might make it nonphysical. Such a Wigner function would not correspond to any physical, positive definite density matrix. This is due to the fact that the operators $\mathbf{R}_{n,\gamma}$ do not commute for different γ's, so noisy measurements may give inconsistent results. Going back from Wigner function to the density matrix using Glauber's formula [18], $\rho = 2\int d^2\gamma(-1)^n W(\gamma^*, \gamma)D(2\gamma)$, one can see in Fig. 3(b), that some diagonal elements of the reconstructed matrix are negative.

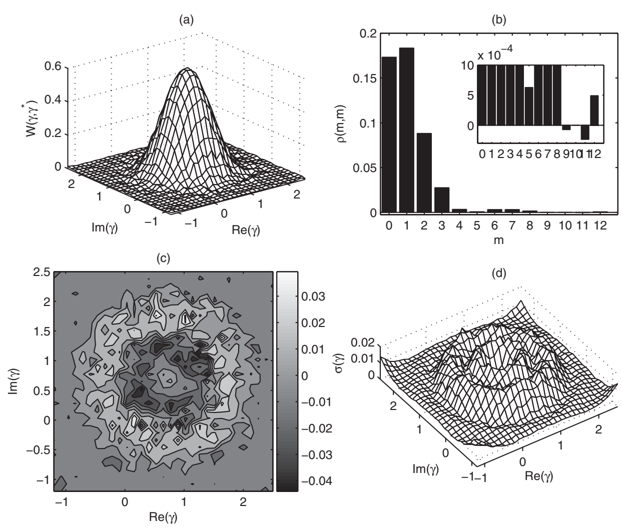

FIG. 3. A reconstruction of the signal coherent state $\alpha = \exp\{i\pi/4\}$. (a) The reconstructed Wigner function; (b) the diagonal elements of the reconstructed density matrix; (c) the difference between the exact and the reconstructed Wigner functions; (d) the variance $\sigma(\gamma, \gamma^*)$. The Wigner function was reconstructed pointwise at 2500 points of the phase plane from $N_r = 10^4$ measurements per point using $N_{\mathrm{it}} = 10^3$ iterations of the EM algorithm. The Hilbert space was truncated at $N_{\mathrm{tr}} = 12$; 30 different values of detector efficiencies were used equidistantly distributed in the interval $[0.1, 0.9]$.

A generic biased tomography scheme addressing some aspects of the quantum systems more efficiently than other aspects has been introduced. Its performance is characterized by quantum analogy of transfer function, which may be further optimized to achieve the desired resolution. This establishes tomography as an objective tool for quantum diagnostics. The recommended approach was demonstrated on a simple, robust and effective quantum tomography scheme using detectors that are only capable of distinguish between the presence and absence of photons.

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