Coherence Current, Coherence Vortex, and the Conservation Law of Coherence

Wei Wang* and Mitsuo Takeda

Department of Information and Communication Engineering, Laboratory for Information Photonics and Wave Signal Processing, The University of Electro-Communications, 1-5-1, Chofugaoka, Chofu, Tokyo 182-8585, Japan

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Introducing scalar and vector densities for a mutual coherence function, we present a new conservation law for optical coherence of scalar wave fields in the form of a continuity equation. This coherence conservation law provides new insights into topological phenomena for the complex coherence function. Some properties related to the newly introduced coherence vector density, such as a circulating coherence current associated with a coherence vortex, are investigated both theoretically and experimentally for the first time.

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Optical fields are inherently of a statistical nature. It has long been recognized that correlations play a fundamental role in describing the statistical properties of light [1]. In particular, the cross correlation between the fluctuating fields at different spacetime points, known as the coherence function, is a quantity of great importance. Since its first introduction by Wolf in 1954 [2], the concept of optical coherence has laid a foundation on which many important problems in statistical optics can be treated in a unified way, and considerable progress has been made due to its theoretical importance and practical interest.

On the other hand, vortices are inherent to many dynamic phenomena as a direct consequence of the spiral motion of a physical quantity [3]. In fluid dynamics, they are fluid vortices originating from a rotary flow velocity field [4]. In optics, they manifest themselves as optical vortices with a spiraling Poynting vector (energy flow) [3,5]. In quantum mechanics, there are vortices of a complex probability wave function related to the Dirac monopole and the Aharonov-Bohm effect stemming from the circular probability current [6]. In all these physical systems, a current, defined as the flow of a certain measurable physical quantity, always obeys a conservation law in the form of continuity equation. Just as the law of conservation of energy applies to any dynamic system, conservation of mass and conservation of momentum also hold in hydrodynamics, and conservation of probability is satisfied in the systems of quantum mechanics. Recently, the existence of coherence vortices or the phase singularities of a complex coherence function has been theoretically predicted [7]. Subsequently, theoretical analyses and experimental investigations [8,9] have been carried out to explore their intriguing characteristics. Among others, the new concept of *coherence current* was introduced, and the coherence current was experimentally observed for a generic coherence vortex [9]. Learning from the history of science on various kinds of currents in different physical systems and their conservation laws, we anticipate that there should be some new conservation law associated with this newly observed coherence current. In this Letter, we will derive, for the first time, a new conservation law in coherence theory, which we will call conservation law of coherence. In formal analogy to energy conservation in scalar wave fields, a scalar density and a vector density are introduced for a mutual coherence function and related by a continuity equation. Some important properties associated with the coherence current are investigated theoretically and experimentally.

In the scalar coherence theory, the propagation of the mutual coherence obeys a pair of wave equations [1]:

$$\nabla^2_{(i)}\Gamma(\mathbf{r}_1,\mathbf{r}_2;t_1,t_2)=\frac{1}{c^2}\frac{\partial^2}{\partial t^2_{(i)}}\Gamma(\mathbf{r}_1,\mathbf{r}_2;t_1,t_2), \quad (1)$$

where the mutual coherence function Γ represents the cross correlation of light vibrations at point $\mathbf{r}_{(i)}$ in space and at instant $t_{(i)}$ in time $(i=1,2)$, $\nabla^2_{(i)}$ and $\partial^2/\partial t^2_{(i)}$ are the differential operations to be performed with respect to the point $\mathbf{r}_i$ and t_i, and c is the speed of light in vacuum. Without loss of generality, we will restrict our discussions to the case for the variation of location $\mathbf{r}_1$ and time t_1 by keeping $\mathbf{r}_2$ and t_2 fixed. Another set of equations that involve $\mathbf{r}_2$ and t_2 can also be derived in a similar way.

If we multiply Eq. (1) for $(i=1)$ by $\partial\Gamma^*/\partial t_1$, use the identity of vector calculus, and add its complex conjugate, we readily obtain the equation

$$\frac{\partial W_\Gamma(\mathbf{r}_1,\mathbf{r}_2;t_1,t_2)}{\partial t_1}+\nabla_1\cdot\mathbf{T}_\Gamma(\mathbf{r}_1,\mathbf{r}_2;t_1,t_2)=0, \quad (2)$$

where

$$W_\Gamma(\mathbf{r}_1,\mathbf{r}_2;t_1,t_2)=\alpha\left[\frac{1}{c^2}\frac{\partial\Gamma^*}{\partial t_1}\frac{\partial\Gamma}{\partial t_1}+\nabla_1\Gamma^*\cdot\nabla_1\Gamma\right], \quad (3)$$

$$\mathbf{T}_\Gamma(\mathbf{r}_1,\mathbf{r}_2;t_1,t_2)=-\alpha\left[\frac{\partial\Gamma^*}{\partial t_1}\nabla_1\Gamma+\frac{\partial\Gamma}{\partial t_1}\nabla_1\Gamma^*\right], \quad (4)$$

and α is a positive constant. From its formal analogy to energy conservation law in scalar wave fields, and from general field-theoretical considerations [10], we note that Eq. (2) is a continuity equation in which the scalar quantity $W_\Gamma(\mathbf{r}_1,\mathbf{r}_2;t_1,t_2)$ may be regarded as representing an energy-density-like quantity of mutual coherence (which we term the *coherenergy density*), and the vector quantity $\mathbf{T}_\Gamma(\mathbf{r}_1, \mathbf{r}_2; t_1, t_2)$ as representing a current-density-like quantity of mutual coherence (which we term the coherence current density). If we integrate Eq. (2) throughout a volume V bounded by a closed surface S and apply the Gauss theorem, we have

$$\frac{d}{dt_1}\iiint_V W_\Gamma(\mathbf{r}_1, \mathbf{r}_2; t_1, t_2) d^3 r_1 + \iint_S \mathbf{T}_\Gamma(\mathbf{r}_1, \mathbf{r}_2; t_1, t_2) \cdot \mathbf{n} dS = 0, \quad (5)$$

where $\mathbf{n}$ denotes the unit outward normal to S. Equation (5) may be given the following interpretation. The rate of increase (decrease) of the coherenergy contained in V is equal to the rate at which the coherenergy enters (leaves) V through the boundary S via the flow of the coherence current. With this interpretation Eq. (5) expresses the conservation law of mutual coherence.

Borrowing from the general theory for vector analysis [11], it is possible to rewrite $\mathbf{T}_\Gamma$ as the sum of two parts based on the Helmholtz decomposition theorem, that is, $\mathbf{T}_\Gamma = -\nabla_1 \varphi_\Gamma + \nabla_1 \times \mathbf{A}_\Gamma$, where φ_Γ is the *coherence scalar potential* introducing the irrotational part of the coherence current and $\mathbf{A}_\Gamma$ is the *coherence vector potential* providing the solenoidal vector component. The significance of this decomposition can be explained as follows. Just as the circulating energy flow of complex scalar wave fields creates a conventional optical vortex [12], the rotational components of $\mathbf{T}_\Gamma$ derived from $\mathbf{A}_\Gamma$ are the sole source for the existence of the phase singularity in the mutual coherence function. Therefore, we define the associated angular coherence momentum density as $\mathbf{M}_\Gamma = \mathbf{r}_1 \times \mathbf{T}_\Gamma / c^2$. Another central role will be played by the vorticity of coherence current density associated with $\mathbf{T}_\Gamma$, namely, $\mathbf{\Omega}_\Gamma = \frac{1}{2}\nabla_1 \times \mathbf{T}_\Gamma$. The vector $\mathbf{\Omega}_\Gamma$ is important because its norm is proportional to the rate of the rotation for coherence current and it points along the coherence vortex line, around which the phase of mutual coherence function increases by 2π in a positive sense with respect to $\mathbf{\Omega}_\Gamma$.

Under the assumption that the statistical properties of the field is stationary and ergodic for time t, the correlation function $\Gamma(\mathbf{r}_1, \mathbf{r}_2; t_1, t_2)$ will then depend on the two time variables only through their difference $t_2 - t_1 = \tau$, and the operator acting on t_1 in all of the equations above may be replaced by an operator acting on τ. In practice, one is often interested in interference effects under quasimonochromatic conditions, where Γ can be rewritten as [1]

$$\Gamma(\mathbf{r}_1, \mathbf{r}_2, \tau) \cong J(\mathbf{r}_1, \mathbf{r}_2) \exp(-j2\pi\bar{\nu}\tau). \quad (6)$$

Here, $J(\mathbf{r}_1, \mathbf{r}_2) = \Gamma(\mathbf{r}_1, \mathbf{r}_2, 0)$ is mutual intensity and $\bar{\nu}$ is the mean frequency of the narrowband light. The expressions for the coherenergy and for the coherence current density in Eqs. (3) and (4) now become, respectively,

$$W_J(\mathbf{r}_1, \mathbf{r}_2) = \alpha(\bar{k})^2[J^* J + (\bar{k})^{-2}\nabla_1 J^* \cdot \nabla_1 J], \quad (7)$$

$$\mathbf{T}_J(\mathbf{r}_1, \mathbf{r}_2) = -j\alpha\bar{k}c[J^*\nabla_1 J - J\nabla_1 J^*] = 2\alpha\bar{k}c|J|^2\nabla_1\theta, \quad (8)$$

where $\bar{k} = 2\pi\bar{\nu}/c$, and we have written $W_J(\mathbf{r}_1, \mathbf{r}_2)$ and $\mathbf{T}_J(\mathbf{r}_1, \mathbf{r}_2)$ in place of $W_\Gamma(\mathbf{r}_1, \mathbf{r}_2; t_1, t_2)$ and $\mathbf{T}_\Gamma(\mathbf{r}_1, \mathbf{r}_2; t_1, t_2)$ to stress that these quantities are now independent of time variables under quasimonochromatic conditions for the stationary field; we term $W_J(\mathbf{r}_1, \mathbf{r}_2)$ as the density of mutual intensity, and $\mathbf{T}_J(\mathbf{r}_1, \mathbf{r}_2)$ as the mutual intensity current density. The differential form of the coherence conservation law in Eq. (2) now becomes $\nabla_1 \cdot \mathbf{T}_J(\mathbf{r}_1, \mathbf{r}_2) = 0$. Therefore, the vector potential of mutual intensity $\mathbf{A}_J$ exists with $\mathbf{T}_J = \nabla_1 \times \mathbf{A}_J$, since $\mathbf{T}_J$ is solenoidal. Meanwhile, the angular coherence momentum density for $\mathbf{T}_J$ may be found by $\mathbf{M}_J = \mathbf{r}_1 \times \mathbf{T}_J / c^2$. The corresponding conservation of the angular momentum in the coherence domain has been reported [9] as the direct extension of the Dennis rule for optical vortex, which is a new version of Keppler's law [12]. Similarly, the vorticity associated with $\mathbf{T}_J$ can be defined as $\mathbf{\Omega}_J = (\nabla_1 \times \mathbf{T}_J)/2$. It follows from this definition that the vorticity is also a solenoidal field due to the fact that $\nabla_1 \cdot \mathbf{\Omega}_J = 0$. Integrating over a finite volume V with the surface S, and using the Gauss theorem, we obtain that $\int_V \nabla \cdot \mathbf{\Omega}_J dV = \int_S \mathbf{\Omega}_J \cdot \mathbf{n} dS = 0$. The line integration of $\mathbf{T}_J$ around a closed curve ℓ, termed *coherence circulation*, gives the flux of $\mathbf{\Omega}_J$ embraced by a surface S:

$$\Phi = \oint_\ell \mathbf{T}_J \cdot d\boldsymbol{\ell} = 2\int_S \mathbf{\Omega}_J \cdot \mathbf{n} dS. \quad (9)$$

Following the convention of fluid dynamics [4], we can introduce some geometrical concepts to characterize the spatial evolution of the coherence vortex. If the closed curve ℓ was chosen as one contour $|J_0|$ for the modulus of the mutual intensity, this contour line is tangential to the vorticity vector serving as the boundary of a small planar area and a collection of these lines can form a coherence vortex tube. After substituting Eq. (8) in Eq. (9), and making use of the definition for phase singularity, $\oint_\ell \nabla\theta \cdot d\boldsymbol{\ell} = 2\pi m$ (m being a topological charge), we obtain

$$\Phi/2 = \int_S \mathbf{\Omega}_J \cdot \mathbf{n} dS = 2\pi\alpha\bar{k}cm|J_0|^2. \quad (10)$$

Equation (10) is analogous to Helmholtz's first theorem in fluid dynamics, which means that the coherence circulation Φ, or the flux of the coherence vorticity, is constant at all cross sections of the coherence vortex tube. Here it should be stressed that the similar relation in Eq. (10) is also applicable for the conventional optical vortex if the closed curve ℓ was chosen as one contour of optical intensity.

As a further approximation, we assume that the mutual intensity J can be expressed as $J(\mathbf{r}_1, \mathbf{r}_2) = I_0\mu(\mathbf{r}_1, \mathbf{r}_2)$, where I_0 is the average intensity and μ is the complex coherence factor. Such is the case in many practical cases of interest when the spatial structure of the source is much larger than its coherence area [1]. By choosing the appropriate constant α in Eq. (8), we can rewrite the coherence current density as $\mathbf{T}_\mu(\mathbf{r}_1, \mathbf{r}_2) = |\mu|^2 \nabla_1 \theta$ with the coherence vorticity given by $\mathbf{\Omega}_\mu = (\nabla_1 \times \mathbf{T}_\mu)/2$. After substituting in Eq. (10), we readily find that

$$\Phi/2 = \int_S \mathbf{\Omega}_\mu \cdot \mathbf{n} dS = \pi |\mu_0|^2, \tag{11}$$

where the line integral for circulation has been performed along the contour line for which $\mu = \mu_0$. Note that, similarly to the definition for coherence time and coherence area [1], the coherence circulation in Eq. (11) is proportional to $|\mu|^2$ rather than $|\mu|$. In the scalar theory of wave field [13], the concept of the intensity has been introduced as the squared modulus of the complex scalar field, which is proportional to the energy power density. In quantum mechanics, the squared modulus determines the probability density for finding the particle since the wave function was regarded as a measure of the probability. Accordingly, we now interpret that $|\mu|^2$ is the intensity of coherence (correlation intensity) because μ is a measure of correlation (degree of similarity) between the two fluctuating fields at different space points. The fact that $|\mu|^2$ is preserved in the newly introduced conservation law of coherence justifies the traditional definition of the coherence time and the coherence area.

To demonstrate the validity of the theory, experiments have been conducted based on the same optical geometry recently proposed for synthesizing and visualizing generic coherence vortices [9]. By moving the position of the reference mirror, we changed the optical path difference Δz between the two arms of the Michelson interferometer and measured the visibility of the fringes by using the Fourier transform method [14]. By changing Δz in 0.05 mm steps, we obtained the distribution of μ along the optical axis and investigated the spatial evolution of $\mathbf{T}_\mu$ enclosing the phase singularity of coherence function.

Figure 1(a) shows an example of the recorded fringes at the position $\Delta z = 0.9$ mm, where μ is represented by the visibility and the shift of the fringes. Figure 1(c) shows the streamlines of coherence current with its labels identifying flow strength, which was calculated from the definition of $\mathbf{T}_\mu$ using the values of μ obtained from the area indicated by the dashed rectangular frame in Fig. 1(a). The fringes recorded for $\Delta z = 1.05$ mm are shown in Fig. 1(b), and Fig. 1(d) is the corresponding coherence current distribution. In all of these figures, we found that the regions of interferograms with high fringe contrast are always labeled with a large value of $\mathbf{T}_\mu$, whose magnitude serves as a measure of the coherence intensity. In analogy to the critical points with zeros of Poynting vector for the optical field [15], the phase of coherence function not only has a phase singularity, which is located at the center of the circular coherence current, but also has a saddle point at point c, where $\nabla_1 \theta$ vanishes with zeros of $\mathbf{T}_\mu$.

As expected, embracing the coherence vortex line, indicated by red crosses, which is determined by the intersection of the zero crossing of real and imaginary parts of

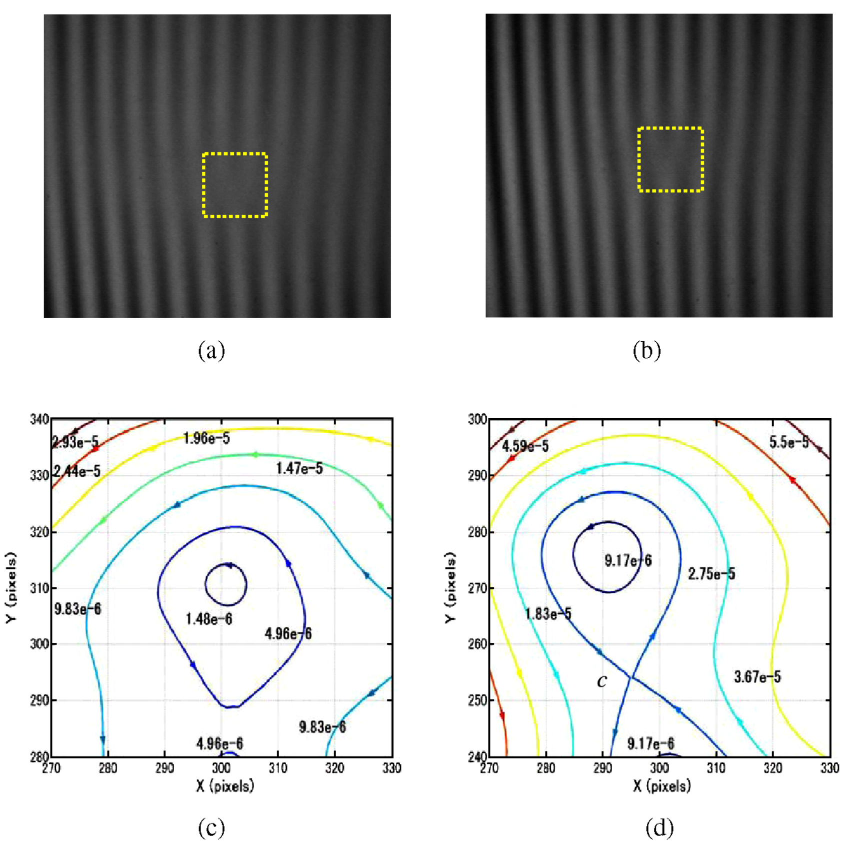

FIG. 1 (color online). The recorded interferograms and the corresponding coherence currents within the area of the dashed rectangular frames for different optical path difference. (a),(c) $\Delta z = 0.9$ mm; (b),(d) $\Delta z = 1.05$ mm. Point c: saddle.

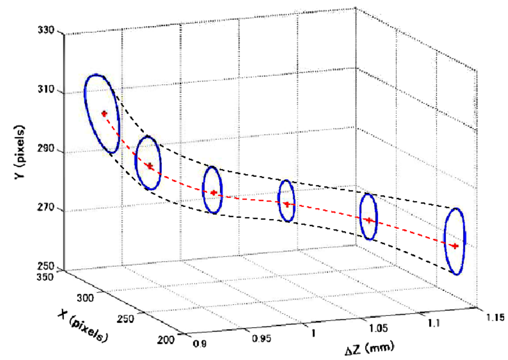

FIG. 2 (color online). A cross-sectional view of a coherence vortex tube.

μ [14], a tubelike structure is observed in Fig. 2, which shows cross sections of the coherence vortex tube whose wall is defined by an equiamplitude surface for $|\mu_0| = 0.01$. Figure 3 shows the relation between the reciprocal of the cross-section area S for coherence vortex tube and its vorticity. With an increase in the value S^{-1}, the magnitude of the coherence vorticity increases linearly. This demonstrates that the circulation around a coherence vortex tube is constant. When the coherence vortex tube comes to a waist, where the tube cross-sectional area is minimal, the average vorticity over that cross section must be maximal, and conversely for the broadening of the tube. A related observation is that vortex tubes cannot terminate in the coherence domain, because, otherwise, the constancy of circulation in Eq. (11) would not be achieved. Therefore, coherence vortex tubes are constrained to form closed loops within the finite spatial volume of the coherence function, or to terminate at infinity.

In summary, we have derived the conservation law of optical coherence for the first time. In terms of a hydrodynamical model, we have theoretically and experimentally investigated the spatial evolution of the newly introduced coherence current vector around the phase singularity of complex coherence function. Furthermore, the proposed method for the analysis of coherence vortices establishes a new relationship between statistical optics and fluid mechanics, which appears to have been regarded as being based on mutually independent disciplines. Further exploration of this may lead to another new field in coherence theory that may be referred to as *coherence dynamics*.

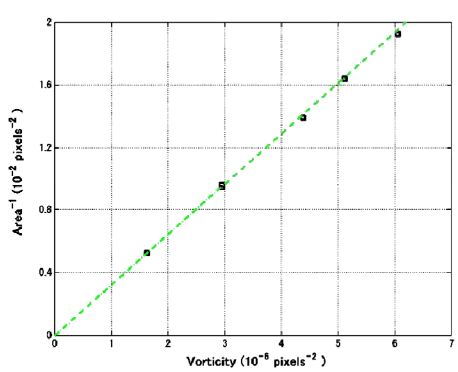

FIG. 3 (color online). The magnitude of vorticity versus the cross-section area of a coherence vortex tube.

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*Electronic address: weiwang@ice.uec.ac.jp

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