3D Quantum Gravity and Effective Noncommutative Quantum Field Theory

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We show that the effective dynamics of matter fields coupled to 3D quantum gravity is described after integration over the gravitational degrees of freedom by a braided noncommutative quantum field theory symmetric under a κ deformation of the Poincaré group.

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One of the most pressing issues in quantum gravity (QG) is the semiclassical regime. In this Letter, we answer this question in the context of matter coupled to 3D gravity. We show how to recover standard quantum field theory (QFT) amplitudes in the no-gravity limit and how to compute the QG corrections.

Let us consider a matter field ϕ coupled to gravity,

$$Z = \int Dg \int D\phi e^{iS[\phi,g]+iS_{GR}[g]}, \tag{1}$$

where g is the spacetime metric, $S_{GR}[g]$ the Einstein gravity action, and $S[\phi, g]$ the action defining the dynamics of ϕ in the metric g. Our goal is to integrate out the QG fluctuations and derive an *effective action* for ϕ taking into account the QG correction:

$$Z = \int D\phi e^{iS_{\rm eff}[\phi]}.$$

We propose to expand the ϕ integration into Feynman diagrams, which depend on the "background" metric g, and to compute the QG effects on these Feynman diagram evaluations:

$$Z = \sum_\Gamma C_\Gamma \int Dg I_\Gamma[g] e^{iS_{GR}[g]} = \sum_\Gamma C_\Gamma \tilde{I}_\Gamma. \tag{2}$$

Finally, we resum these deformed Feynman diagrams to identify the effective action $S_{\rm eff}[\phi]$ taking into account the QG corrections to the matter dynamics. Here, we prove that this program can be explicitly realized for 3D quantum gravity. The resulting effective matter theory is a noncommutative field theory invariant under the κ deformed Poincaré group. The deformation parameter κ is simply related to the Newton constant for gravitation $\kappa = 4\pi G$. All technical proofs can be found in [1].

In a first order formalism, Riemannian 3D gravity is described in the term of a frame field $e^i_\mu dx^\mu$ and a spin connection $\omega^i_\mu dx^\mu$, both valued in the Lie algebra $\mathfrak{so}(3)$. Indices i and μ run from 0 to 2. The action is defined as

$$S[e, \omega] = \frac{1}{16\pi G} \int e^i \wedge F_i[w], \tag{3}$$

where $F \equiv d\omega + \omega \wedge \omega$ is the curvature tensor of the 1-form ω. The equations of motion for pure gravity impose that the connection is flat and the torsion vanishes,

$$F[\omega] = 0, \qquad T[\omega, e] = d_\omega e = 0. \tag{4}$$

This is actually a topological field theory. Particles are introduced as topological defects [2]. Spinless particles are a source of curvature (the spin introduces torsion):

$$F^i[\omega] = 4\pi G p^i \delta(x).$$

Outside the particle, the spacetime remains flat and the particle creates a conical singularity with a deficit angle related to the particle's mass [3]:

$$\theta = \kappa m. \tag{5}$$

This deficit angle describes the feedback of the particle on the spacetime geometry. Since θ is obviously bounded by 2π, particles have a maximal allowed mass $m_P = (2G)^{-1}$. Note that the Planck mass m_P in three dimensions does not depend on the Planck constant unlike the Planck length $l_P = \hbar m_P^{-1} \sim \hbar G$. This feature is specific to 3D QG and does not apply to the 4D theory.

The spin foam quantization of 3D gravity is given by the Ponzano-Regge model [4], which was the first ever written QG model. It is a discretization of the continuum path integral, $Z = \int De D\omega e^{iS[e,\omega]}$. Since the theory is topological, the discretization actually provides an exact quantization. Considering a triangulation Δ of a 3D manifold $\mathcal{M}$ and a graph $\Gamma \subset \Delta$, we insert particles with deficit angles θ_e for all edges $e \in \Gamma$ of the graph. The partition function is defined as the product of weights associated with the edges and the tetrahedra:

$$I_\Delta[\Gamma] = \sum_{\{j_e\}} \prod_{e \notin \Gamma} d_{j_e} \prod_{e\in\Gamma} K_{\theta_e}(j_e) \prod_t \begin{Bmatrix} j_{e_1} & j_{e_2} & j_{e_3} \\ j_{e_4} & j_{e_5} & j_{e_6} \end{Bmatrix}, \tag{6}$$

where we sum over all assignments of SO(3) representation $j_e \in \mathbb{N}$ to the edges of Δ. $d_j = (2j + 1)$ is the dimension of the j representation and we associate a $\{6j\}$ symbol with each tetrahedron. $h_\theta = \exp(i\theta\sigma_3)$ is in the U(1) subgroup and we define the weight:

$$K_\theta(j) = \frac{i}{2\kappa^2} \frac{e^{-id_j(\theta - i\epsilon)}}{\cos\theta}, \qquad \mathrm{Re} K_\theta = \frac{\cos\theta}{2\kappa^2 \sin\theta} \chi_j(\theta),$$

where $\epsilon > 0$ is a regulator and $\chi_j(\theta)$ the trace of h_θ in the j

representation. K_θ defines the insertion of a Feynman propagator, while ReK_θ gives a Hadamard propagator and leads back to the same partition function as in [5].

The partition function has a dual formulation in terms of SO(3) group elements attached to the faces $f \in \Delta$:

$$I_\Delta[\Gamma, \theta_e] = \int \prod_f dg_f \prod_{e\in\Gamma} \tilde{K}_{\theta_e}(g_e) \prod_{e\notin\Gamma} \delta(g_e),$$
$$\tilde{K}_{\theta_e}(h_\phi) = \frac{i}{\kappa^2} \frac{1}{(\sin^2\phi - \sin^2\theta_e + i\epsilon)} = \sum_j K_\theta(j)\chi_j(\phi), \tag{7}$$

where g_e is the oriented product $\prod_{\partial f \ni e} g_f$ and the function $\tilde{K}_\theta(g)$ is invariant under conjugation. Using the real part of K_θ leads to replacing $\tilde{K}_\theta$ by the distribution $\delta_\theta(g)$, which fixes the rotation angle of g to θ,

$$\int_{\mathrm{SO}(3)} dg f(g)\delta_\theta(g) = \int_{\mathrm{SO}(3)/\mathrm{U}(1)} dx f(x h_\theta x^{-1}).$$

$I_\Delta[\Gamma]$ is independent of the triangulation Δ and depends only on the topology of $(\mathcal{M}, \Gamma)$. It is finite after suitable gauge fixing of the diffeomorphism symmetry [6], which removes redundancies in the product of δ functions. Then for a trivial topology $\mathcal{M} = [0, 1] \times \Sigma_2$, I_Δ is the projector onto the physical states, that is, the space of flat connections on Σ_2 [7]. Moreover, this quantization scheme has been shown to be equivalent to the Chern-Simons quantization [8]. Finally, the large j asymptotics of the $\{6j\}$ symbols are related to the discrete Regge action for 3D gravity [9].

We have defined a purely algebraic quantum gravity amplitude $I_\Delta[\Gamma]$. The Newton constant G only appears as a unit to translate the algebraic quantities j, θ into the physical length $l = jl_P = j\hbar G$ and the physical mass $m = \theta/\kappa = \theta/4\pi G$.

The essential point is that the QG amplitudes $I_\Delta[\Gamma]$ are the Feynman diagram evaluations of a noncommutative field theory. Let us first consider a trivial topology $\mathcal{M} \sim \mathcal{S}^3$ with Γ planar. In this case, we can get rid of the triangulation dependence and rewrite $I_\Gamma \equiv I_\Delta[\Gamma]$ as [1]

$$I_\Gamma = \int \prod_{v\in\Gamma} \frac{d^3X_v}{8\pi\kappa^3} \int \prod_{e\in\Gamma} dg_e \tilde{K}_{\theta_e}(g_e) \prod_{v\in\Gamma} e^{(1/2\kappa)\mathrm{tr}(X_v G_v)}. \tag{8}$$

The integral is over one copy of $\mathfrak{so}(3) \sim \mathbb{R}^3$ for each vertex $X_v \equiv X_v^i \sigma_i$ and one copy of SO(3) for each edge. We define at each vertex v, the ordered product of the edge group elements meeting at v,

$$G_v = \overrightarrow{\prod_{e\supset v}} g_e^{\epsilon_v(e)}, \tag{9}$$

$\epsilon_v(e) = \pm 1$ depending on whether the edge is incoming or outgoing at v. The kernel $\tilde{K}_\theta$ defines the Feynman propagator and is given by

$$\tilde{K}_\theta(g) = \int_{\mathbb{R}^+} dT e^{iT[P^2(g) - (\sin\kappa m/\kappa)^2]}, \tag{10}$$

with $2i\vec{P}(g) \equiv \mathrm{tr}(g\vec{\sigma})$ the projection of g on Pauli matrices. Changing the integration range from $\mathbb{R}^+$ to $\mathbb{R}$, we would obtain the Hadamard function δ_θ instead of $\tilde{K}_\theta$. To further simplify this expression, we introduce a noncommutative $\star$ product on $\mathbb{R}^3$ such that

$$e^{(1/2\kappa)\mathrm{tr}(Xg_1)} \star e^{(1/2\kappa)\mathrm{tr}(Xg_2)} = e^{(1/2\kappa)\mathrm{tr}(Xg_1g_2)}. \tag{11}$$

Using the parametrization of SO(3) group elements,

$$g = (P_4 + \iota\kappa P^i\sigma_i), \qquad P_4^2 + \kappa^2 P^i P_i = 1, \qquad P_4 \geq 0,$$

the $\star$ product deforms the composition of plane waves,

$$e^{\iota(\vec{P}_1 \oplus \vec{P}_2)\cdot\vec{X}} = e^{\iota\vec{P}_1\cdot\vec{X}} \star e^{\iota\vec{P}_2\cdot\vec{X}}, \tag{12}$$

$$\vec{P}_1 \oplus \vec{P}_2 = \sqrt{1-\kappa^2|\vec{P}_2|^2}\vec{P}_1 + \sqrt{1-\kappa^2|\vec{P}_1|^2}\vec{P}_2 - \kappa\vec{P}_1 \times \vec{P}_2, \tag{13}$$

with $\times$ the 3D vector cross product. To define the $\star$ product on all functions, we introduce a new *group Fourier transform F*: $C(\mathrm{SO}(3)) \to C_\kappa(\mathbb{R}^3)$ mapping functions on the group SO(3) to functions on $\mathbb{R}^3$ with momenta bounded by $1/\kappa$:

$$\phi(X) = \int dg \tilde{\phi}(g) e^{(1/2\kappa)\mathrm{tr}(Xg)}. \tag{14}$$

The inverse group Fourier transform is explicitly written

$$\tilde{\phi}(g) = \int_{\mathbb{R}^3} \frac{d^3X}{8\pi\kappa^3} \phi(X) \star e^{(1/2\kappa)\mathrm{tr}(Xg^{-1})}$$
$$= \int_{\mathbb{R}^3} \frac{d^3X}{8\pi\kappa^3} \phi(X)\sqrt{1-\kappa^2P^2(g)} e^{(1/2\kappa)\mathrm{tr}(Xg^{-1})}. \tag{15}$$

Under this Fourier transform, the $\star$ product is dual to the group convolution product. Finally, F is an isometry between $L^2(\mathrm{SO}(3))$ and $C_\kappa(\mathbb{R}^3)$ equipped with the norm

$$\|\phi\|_\kappa^2 = \int \frac{dX}{8\pi\kappa^3} \phi \star \phi(X). \tag{16}$$

Using this $\star$ product, the amplitude (8) reads

$$I_\Gamma = \int \prod_{v\in\Gamma} \frac{dX_v}{8\pi\kappa^3} \prod_{e\in\Gamma} dg_e \tilde{K}_{\theta_e}(g_e) \prod_{v\in\Gamma} (\bigstar_{v\in\Gamma} e^{[\epsilon_v(e)/2\kappa]\mathrm{tr}(X_v g_e)}). \tag{17}$$

Let us now restrict to the case where we have particles of only one type so all masses are taken equal, $m_e \equiv m$, and consider the sum over trivalent graphs:

$$\sum_{\Gamma\ \mathrm{trivalent}} \frac{\lambda^{|v_\Gamma|}}{S_\Gamma} I_\Gamma, \tag{18}$$

where λ is a coupling constant, $|v_\Gamma|$ is the number of vertices of Γ, and S_Γ is the symmetry factor of the graph. Remarkably, this sum can be obtained from the perturbative expansion of a noncommutative field theory given explicitly by

$$S = \int \frac{d^3x}{8\pi\kappa^3} \Big[\frac{1}{2}(\partial_i\phi \star \partial_i\phi)(x) - \frac{1}{2}\frac{\sin^2 m\kappa}{\kappa^2}(\phi\star\phi)(x) + \frac{\lambda}{3!}(\phi\star\phi\star\phi)(x) \Big], \tag{19}$$

where the field ϕ is in $C_\kappa(\mathbb{R}^3)$. Its momentum has support in the ball of radius κ^{-1}. We can write this action in momentum space,

$$S(\phi) = \frac{1}{2}\int dg\left(P^2(g) - \frac{\sin^2\kappa m}{\kappa^2}\right)\tilde{\phi}(g)\tilde{\phi}(g^{-1}) + \frac{\lambda}{3!} \times \int dg_1 dg_2 dg_3 \delta(g_1 g_2 g_3)\tilde{\phi}(g_1)\tilde{\phi}(g_2)\tilde{\phi}(g_3). \tag{20}$$

This is our effective field theory describing the dynamics of the matter field after integrating out the gravitational sector. This noncommutative field theory action is symmetric under a κ deformed action of the Poincaré group. Calling Λ the generators of Lorentz transformations and $T_{\vec{a}}$ the generators of translations, the action of these generators on one-particle states is undeformed:

$$\Lambda\tilde{\phi}(g) = \tilde{\phi}(\Lambda g \Lambda^{-1}) = \tilde{\phi}(\Lambda P(g)), \tag{21}$$

$$T_{\vec{a}}\tilde{\phi}(g) = e^{i\vec{P}(g)\vec{a}}\tilde{\phi}(g). \tag{22}$$

The nontrivial deformation of the Poincaré group appears at the level of multiparticle states, and only the action of the translations is deformed:

$$\Lambda\tilde{\phi}(P_1)\tilde{\phi}(P_2) = \tilde{\phi}(\Lambda P_1)\tilde{\phi}(\Lambda P_2), \tag{23}$$

$$T_{\vec{a}}\tilde{\phi}(P_1)\tilde{\phi}(P_2) = e^{i\vec{P}_1 \oplus \vec{P}_2 \vec{a}}\tilde{\phi}(P_1)\tilde{\phi}(P_2). \tag{24}$$

It is straightforward to derive the Feynman rules from the action (21) (see Fig. 1). The effective Feynman propagator is the group Fourier transform of $\tilde{K}_\theta(g)$,

$$K_m(X) = i\int dg \frac{e^{(1/2\kappa)\mathrm{tr}(Xg)}}{P^2(g) - (\frac{\sin\kappa m}{\kappa})^2}. \tag{25}$$

The effect of quantum gravity is twofold. First, the mass gets renormalized $m \to \sin\kappa m/\kappa$. Second, the momentum space is no longer the flat space but the homogeneously curved space $S^3 \sim \mathrm{SO}(3)$. This reflects that the momentum is bounded $|P| < 1/\kappa$.

At the interaction vertex the momentum addition becomes nonlinear with a conservation rule $P_1 \oplus P_2 \oplus P_3 = 0$, which implies a nonconservation of momentum $P_1 + P_2 + P_3 \neq 0$. Intuitively, part of the energy involved in a collision process is absorbed by the gravitational field: gravitational effects cannot be ignored at high energy. This effect, which is stronger at high momenta and for noncollinear momenta, prevents the total momenta from being larger than the Planck energy.

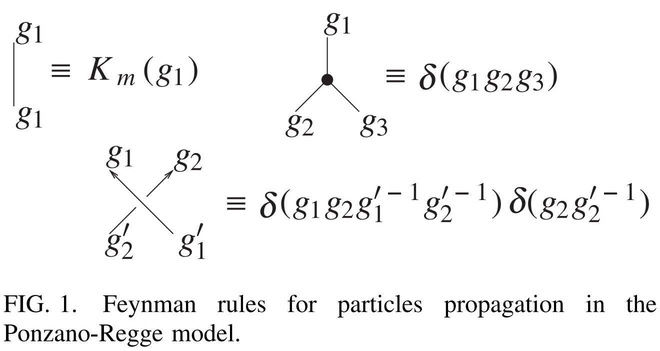

FIG. 1. Feynman rules for particles propagation in the Ponzano-Regge model.

A last subtlety of the Feynman rules is the evaluation of nonplanar diagrams. A careful analysis of I_Γ shows that we have a nontrivial *braiding*: for each crossing of two edges, we associate a weight $\delta(g_1 g_2 g_1'^{-1} g_2'^{-1})\delta(g_2 g_2'^{-1})$ (see Fig. 1). This reflects a nontrivial statistics where the Fourier modes of the fields obey the exchange relation:

$$\tilde{\phi}(g_1)\tilde{\phi}(g_2) = \tilde{\phi}(g_2)\tilde{\phi}(g_2^{-1}g_1 g_2), \tag{26}$$

which is naturally determined by our choice of star product. Indeed, let us look at the product of two identical fields:

$$\phi \star \phi(X) = \int dg_1 dg_2 e^{(1/2\kappa)\mathrm{tr}(Xg_1g_2)}\tilde{\phi}(g_1)\tilde{\phi}(g_2), \tag{27}$$

Under change of variables $(g_1, g_2) \to (g_2, g_2^{-1}g_1g_2)$, the star product reads

$$\phi \star \phi(X) = \int dg_1 dg_2 e^{(1/2\kappa)\mathrm{tr}(Xg_1g_2)}\tilde{\phi}(g_2)\tilde{\phi}(g_2^{-1}g_1g_2). \tag{28}$$

The identification of the Fourier modes of $\phi \star \phi(X)$ leads to the exchange relation (26). This braiding was first proposed in [10] for two particles coupled to 3D QG and then computed in the Ponzano-Regge model in [5]. It is encoded into a braiding matrix

$$R\tilde{\phi}(g_1)\tilde{\phi}(g_2) = \tilde{\phi}(g_2)\tilde{\phi}(g_2^{-1}g_1g_2). \tag{29}$$

This is the R matrix of the κ deformation of the Poincaré group [10]. Such field theories with nontrivial braided statistics are simply called braided noncommutative field theories and were first introduced in [11].

Finally, the $\star$ product induces a noncommutativity of spacetime and a deformation of phase space:

$$\begin{aligned} [X_i, X_j] &= i\kappa\epsilon_{ijk}X_k, \\ [X_i, P_j] &= i\sqrt{1-\kappa^2P^2}\delta_{ij} - i\kappa\epsilon_{ijk}P_k. \end{aligned} \tag{30}$$

This noncommutativity reflects the fact that momentum space is curved. Indeed the coordinates X are realized as right invariant derivations on momentum space and derivations of a curved manifold do not commute. Moreover, this noncommutativity being related to having bounded momenta implies the existence of a minimal length scale accessible in the theory. Indeed defining the noncommutative δ function $\delta_0 \star \phi(X) = \phi(0)\delta_0(X)$, we compute

$$\delta_0(X) = 2\kappa\frac{J_1(\frac{|X|}{\kappa})}{|X|}, \tag{31}$$

with J_1 the first Bessel function. It is clear that $\delta_0(X)$ is concentrated around $X = 0$ but has a nonzero width.

Using this formalism, one can compute the QG effects order by order in κ. The zeroth order is defined by the no-gravity limit $\kappa \to 0$. Starting from either the spin foam amplitude I_Γ given by (6) or the Feynman evaluations (17), one can show that the limit $\kappa \to 0$ is exactly given by the Feynman evaluations of the usual commutative QFT:

$$I^0_\Delta[\Gamma, m_e] \equiv \lim_{\kappa\to 0} \kappa^{3|e_\Gamma|} I_\Delta[\Gamma, \theta_e] = \int_{\mathbb{R}^3} \prod_{f\in\Delta} d^3\vec{p}_f \prod_{e\in\Gamma} \frac{i}{2\pi(p_e^2 - m_e^2)} \prod_{e\in\Delta\setminus\Gamma} \delta(\vec{p}_e), \quad (32)$$

where $|e_\Gamma|$ is the number of edges of the graph Γ, $\vec{p}_f \in \mathbb{R}^3$ are variables attached to the faces of Δ, and $\vec{p}_e = \sum_{f\supset e}\vec{p}_f$. Moreover, since physical lengths and masses are defined in κ units, $l = \kappa j$ and $m = \theta/\kappa$, taking $\kappa \to 0$ corresponds to $j \to \infty$ and $\theta \to 0$. For $\theta \sim 0$, the group multiplication on SO(3) becomes Abelian at first order in κ. More precisely, we prove in [1] that the no-gravity limit of the Ponzano-Regge model is actually the topological state sum based on the Abelian group $\mathbb{R}^3$. This shows that the usual Feynman evaluations of QFT in three dimensions can be generically written as amplitudes of a topological theory.

Up to now, we have worked in the Riemannian context. All the previous constructions and results can be straightforwardly extended to the Lorentzian theory. The Lorentzian version of the Ponzano-Regge model is expressed in terms of the $\{6j\}$ symbols of the noncompact group SO(2, 1) [12]. Holonomies around particles are SO(2, 1) group elements parametrized as $g = P_4 + i\kappa P_i\tau^i$ with $P_4^2 + \kappa^2 P_i P^i = 1$ and $P_4 \geq 0$, with the metric $(+--)$ and the $\mathfrak{su}(1,1)$ Pauli matrices, $\tau_0 = \sigma_0$, $\tau_{1,2} = i\sigma_{1,2}$. Massive particles correspond to the $P_iP^i > 0$ sector. They are described by elliptic group elements, $P_4 = \cos\theta$, $\kappa|P| = \sin\theta$. The deficit angle is given by the mass, $\theta = \kappa m$. All the mathematical relations of the Riemannian theory are translated to the Lorentzian framework by changing the signature of the metric. The propagator remains given by the formula (10). The momentum space is now the anti–de Sitter space $\mathrm{AdS}^3 \sim \mathrm{SO}(2,1)$. The addition of momenta is deformed according to the formula (13). We similarly introduce a group Fourier transform $F: C(\mathrm{SO}(2,1)) \to C_\kappa(\mathbb{R}^3)$ and a $\star$ product dual to the convolution product on SO(2, 1). Finally, we derive the effective noncommutative field theory with the same expression (19) as in the Riemannian case.

To summarize, we have shown that the 3D quantum gravity amplitudes, defined through the Ponzano-Regge spin foam model, are actually the Feynman diagram evaluations of a (braided) noncommutative QFT. This effective field theory describes the dynamics of the matter field after integration of the gravitational degrees of freedom. The theory is invariant under a κ deformation of the Poincaré algebra, which acts nontrivially on many-particle states. This is an explicit realization of a QFT in the framework of deformed special relativity (see, e.g., [13]), which implements from first principles the original idea of Snyder [14] of using a curved momentum space to regularize the Feynman diagrams.

The formalism can naturally take into account a nonzero cosmological constant Λ. The model is based on $\mathcal{U}_q(\mathrm{SU}(2))$ and its Feynman rules are given in [1].

A natural question concerns the unitarity of our noncommutative quantum field theory since the noncommutativity affects time [15]. We *a priori* do not expect a unitary theory: since we have integrated out the gravity degrees of freedom, we expect ghosts to appear at the Planck energy $m_P = 1/\kappa \sim 1/G$.

Finally, the present results suggest an extension to four dimensions. The standard 4D QFT Feynman graphs would be expressed as expectation values of a 4D topological spin foam model (see, e.g., [16,17]). That model would provide the semiclassical limit of QG and be identified as the zeroth order of an expansion in term of the inverse Planck mass κ of the full QG spin foam amplitudes. QG effects would then appear as deformations of the Feynman graph evaluations, and QG corrections to the scattering amplitudes could be computed order by order in κ.

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