New Formulation of the Equation for Synchrotron Oscillations

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We develop an orbit theory for synchrotron oscillations using the orbit length, s, as an independent variable. This is commonplace for static magnetic fields (storage rings). We extend this to the case of adiabatically varying magnetic fields (synchrotrons). Contrary to conventional treatments, betatron acceleration terms appear in both the energy and phase equations. We derive one-turn difference equations in the linear and adiabatic approximations. By a smooth approximation instead of the traveling-wave approximation, and by combining the two equations, we obtain a differential equation where the betatron acceleration terms are canceled. This equation is an extension of McMillan's equation to the case of strong-focusing synchrotrons.

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All of the devices in circular accelerators are placed at fixed positions along the circumference of the machine. Observations are also made at fixed positions. Thus, the orbit length, s, is a natural independent variable, and the (arrival) time and the energy are dependent (canonical) variables. This description (s description) is commonplace for synchrotron oscillations in the static case (constant magnetic fields). However, the time, t, is usually used for synchrotron oscillations in changing magnetic fields. In this description (t description), it is difficult to study the localized nature of rf cavities and impedance sources. We are forced to use a traveling-wave approximation. On the other hand, in the standing-wave picture (s description), which is more physical, the localized objects are expressed simply by periodic delta functions.

Another confusion in the existing literature is betatron acceleration. This term (the $\dot{B}$ term) is sometimes neglected, where $\dot{B}$ means the partial derivative with respect to the time of the vertical magnetic induction, B, but this must be included. This problem has been studied by various authors. Among them, Kolomensky and Lebedev [1] started from the s description for the energy equation with the betatron acceleration term. By using the traveling-wave approximation and changing the independent variable from s to t, they showed that the $\dot{B}$ term disappears in the energy equation.

In the present theory, we start from the s description for both the energy and the phase equations. Here, we ignore the nonlinear terms such as $\dot{B}^2$, etc. (adiabatic approximation). We further make linear approximations for longitudinal and transverse coordinate variables except in the rf-acceleration term. Then, we find that the $\dot{B}$ terms appear in the phase and energy equations. We can transform these differential equations to difference (one-turn mapping) equations in the adiabatic approximation. We then approximate the difference equations by differential equations using a smooth approximation instead of the traveling-wave approximation. When we combine these equations, we find that the $\dot{B}$ terms cancel each other under adiabatic conditions. This combined equation is an extension of the one given by McMillan [2] to strong-focusing synchrotrons using the s description. The present Letter is based on a paper [3] by the present author. Here, we clarify some missing points in Ref. [3]. The main additions are the difference (mapping) equations, the smooth approximation instead of the traveling-wave approximation, and the clarification of the replacement of $D(s)/\rho(s)$ by α, where $D(s)$ is the dispersion function, $\rho(s)$ is the radius of curvature of the design orbit, and α is the momentum compaction factor. These additions will make the theory applicable to strong-focusing synchrotrons.

In the s description, the (arrival) time, t, and minus the energy, $-E$, are canonical variables. We first make a canonical transformation from t to τ by the relation $t = t_0 + \tau$, where t_0 is the arrival time of a synchronous particle, $t_0(s) = \int^s ds/v_0$, v_0 is the velocity of the synchronous particle, and $\tau(s)$ is the time delay of an arbitrary particle. We put subscript 0 to variables of the synchronous particle in this Letter. We then make the second canonical transformation from $(\tau, -E)$ to $(\tau, -\Delta E)$ by the relation $E = E_0 + \Delta E$, where E_0 is the energy of the synchronous particle and ΔE is the energy error. Although the equations of motion can be derived from a Hamiltonian, we can obtain them from physical considerations if we pay due attention to the canonical natures of the variables. We describe this simplified approach though the equations are checked by a Hamiltonian formalism.

The energy equation is

$$\frac{d\Delta E}{ds} = eV\delta_p(s - s_c)\{\sin\phi - \sin\phi_0\} + e\dot{B}x, \quad (1)$$

where eV is the peak energy gain by rf cavities, ϕ is the rf phase, s_c is the position of the rf cavity, δ_P is the periodic δ function, and x is the horizontal displacement. In this Letter, the dot means a partial derivative with respect to time. The betatron acceleration term is derived locally from the vector potential $\vec{A}$ by the relation $\vec{E} = -\partial\vec{A}/\partial t$, where $\vec{E}$ is the electric field strength. The scalar potential can be put to be zero in a charge-free region in vacuum. We ignore nonlinear terms such as edge effects and assume a two-dimensional field, where $\vec{A}$ is independent of s. Then, the vector potential is easily obtained. This gives the betatron acceleration term in Eq. (1) in the linear approximation, where $\dot{B}$ is independent of s inside each magnet. Now, since only the combination $\dot{B}/B$ appears in later developments, the value of $\dot{B}$ can vary from magnet to magnet in proportion to the value of B.

The time equation is derived by simple geometrical considerations. Keeping only linear terms, we obtain after several steps

$$\frac{d\tau}{ds} = \frac{1}{v_0}\left(\frac{x}{\rho} - \frac{1}{\beta_0^2\gamma_0^2}\frac{\Delta E}{E_0}\right), \qquad (2)$$

where β_0 and γ_0 are the Lorentz factors of the synchronous particle.

Now, x is decomposed as

$$x = x_\beta + D\left(\frac{\Delta E}{\beta_0^2 E_0} - \frac{\Delta B}{B_0}\right) + x_{\mathrm{co}}, \qquad (3)$$

where x_β is the coordinate of betatron oscillations, ΔB is the field error, and x_{co} denotes a closed-orbit distortion driven by errors. Usually, only the $\Delta E/E_0$ term in Eq. (3) is kept for synchrotron oscillations, but the ΔB term is also important for symplectic descriptions. Different particles pass through a fixed point, s, at different times and feel different magnetic field strengths. Thus,

$$\Delta B(t_0 + \tau) = \Delta B(t_0) + \dot{B}(t_0)\tau. \qquad (4)$$

Since the $\Delta B(t_0)$ term does not contain any canonical variable, it affects only a closed orbit, but the $\dot{B}$ term is important. If this term is neglected, the necessary condition for a symplectic description,

$$\frac{\partial \Delta E'}{\partial \Delta E} + \frac{\partial \tau'}{\partial \tau} = 0, \qquad (5)$$

is not satisfied. Here, the primes denote differentiation with respect to s. We can prove this condition by inserting Hamilton's equations of motion into Eq. (5). The x_β term shows a synchrobetatron coupling. We neglect this and the closed-orbit distortions.

Inserting Eq. (3) into Eqs. (1) and (2), we obtain the following energy and time equations:

$$\frac{d\Delta E}{ds} = e\dot{B}D\frac{\Delta E}{\beta_0^2 E_0} + eV\delta_p(s - s_c)\{\sin\phi - \sin\phi_0\}, \qquad (6)$$

$$\frac{d\tau}{ds} = \frac{1}{v_0}\left\{\left(\frac{D}{\rho} - \frac{1}{\gamma_0^2}\right)\frac{\Delta E}{\beta_0^2 E_0} - \frac{D\dot{B}}{\rho B_0}\tau\right\}. \qquad (7)$$

We note that $(e\dot{B})/(\beta_0^2 E_0) = \dot{B}/(v_0 B_0 \rho)$ from the well-known relation $p_0 = eB_0\rho$, where p_0 is the momentum of the synchronous particle. Because of the presence of $\dot{B}$, ρ is meaningful only inside the magnets and is equal to the radius of curvature of the central orbit. Thus Eq. (5) is satisfied. The above results were already obtained by the present author [3]. The problem that we wish to clarify here is the validity of the replacement of D/ρ by its one-turn average, α. This replacement is exact for weak-focusing synchrotrons. In strong-focusing synchrotrons, more explanation is necessary. Actually, we will show that this averaging is valid in the adiabatic approximation.

We can simply solve Eq. (6) outside the rf cavity and obtain

$$\Delta E(s) = \Delta E_i \exp\left(a\int_0^s \frac{D(s')}{\rho(s')}ds'\right), \qquad (8)$$

where

$$a = \dot{B}/(v_0 B_0) \qquad (9)$$

is a constant in the present adiabatic approximation. We put $s_c = 0$ without loss of generality. The integral is performed within the magnets because of the presence of $\dot{B}$. We put $\rho = \rho(s)$ in the integral (8) for the sake of clarity. Here, ΔE_i is the value at the exit of the cavity. We insert Eq. (8) into Eq. (7), which results in a first-order linear inhomogeneous differential equation,

$$\frac{d\tau}{ds} = -\frac{D\dot{B}}{v_0\rho B_0}\tau + \frac{1}{v_0}\left(\frac{D}{\rho} - \frac{1}{\gamma_0^2}\right)\frac{\Delta E_i}{\beta_0^2 E_0} \times \exp\left(a\int_0^s \frac{D(s')}{\rho(s')}ds'\right). \qquad (10)$$

We can solve Eq. (10) simply by the method of variable constant,

$$\tau(s) = \exp[-b\tilde{\alpha}(s)]\left[\tau_i + \int_0^s\left\{\frac{1}{v_0}\left(\frac{D(s')}{\rho(s')} - \frac{1}{\gamma_0^2}\right)\frac{\Delta E_i}{\beta_0^2 E_0} \times \exp[2b\tilde{\alpha}(s')]\right\}ds'\right], \qquad (11)$$

where we put

$$2\pi R\tilde{\alpha}(s) = \int_0^s D(s')/\rho(s')ds', \qquad (12)$$

$$b = 2\pi R a = 2\pi R\dot{B}/(v_0 B_0). \qquad (13)$$

Here, $\tilde{\alpha}(2\pi R) = \alpha$, and R is the average radius of the synchrotron.

We now expand the exponentials in Eqs. (8) and (11) into a Taylor series and keep up to the first-order terms in $\dot{B}$. We then obtain for one turn

$$\tau_f = \tau_i + \frac{2\pi R}{v_0}\left(\alpha - \frac{1}{\gamma_0^2}\right)\frac{\Delta E_i}{\beta_0^2 E_0} - b\alpha\tau_i \qquad (14)$$

and

$$\Delta E_f = \Delta E_i + eV\{\sin(\phi_0 + \Delta\phi) - \sin\phi_0\} + b\alpha\Delta E_i, \tag{15}$$

where the subscript f denotes the value after one turn. We use the following integrals:

$$\int_0^{2\pi R} \tilde{\alpha}(s)\frac{D(s)}{\rho(s)}ds = \frac{1}{2}\alpha^2 \tag{16}$$

and

$$\int_0^{2\pi R} \tilde{\alpha}(s)ds = \pi\alpha R. \tag{17}$$

In Eq. (17), we use the fact that $D(s)/\rho(s)$ is periodic with a period of $2\pi R$. Thus, we obtain difference (one-turn map) equations [(14) and (15)] for synchrotron oscillations for the case where the magnetic fields are changing with time (synchrotrons).

We now approximate these difference equations by differential equations (smooth approximation) as

$$\frac{d\Delta E}{ds} \approx \frac{\Delta E_f - \Delta E_i}{2\pi R}, \tag{18}$$

$$\frac{d\tau}{ds} \approx \frac{\tau_f - \tau_i}{2\pi R}. \tag{19}$$

Then, by changing the independent variable from s to $\theta(s = R\theta)$, we obtain

$$\frac{d\Delta E}{d\theta} = \frac{eV}{2\pi}\{\sin(\phi_0 + \Delta\phi) - \sin\phi_0\} + \frac{\alpha\dot{B}}{\omega_0 B_0}\Delta E, \tag{20}$$

$$\frac{d\tau}{d\theta} = \frac{\eta\Delta E}{\omega_0\beta_0^2 E_0} - \frac{\alpha\dot{B}}{\omega_0 B_0}\tau, \tag{21}$$

where $\eta = \alpha - 1/\gamma_0^2$ and ω_0 is the angular revolution frequency. Combining Eqs. (20) and (21), we obtain an equation for τ,

$$\frac{d}{d\theta}\left(\frac{\omega_0\beta_0^2 E_0}{\eta}\frac{d\tau}{d\theta}\right) = \frac{eV}{2\pi}(\sin\phi - \sin\phi_0), \tag{22}$$

where the second-order term $\dot{B}^2$ is consistently omitted. We note that the $\dot{B}$ terms appear only in the second or higher order terms.

The arrival time at a fixed point has a strict physical significance, but we usually use the rf phase. Here, we make a brief comment on the rf-phase angle, $\phi = \phi_0 + \Delta\phi$. In the standing-wave picture, the particles feel an rf field only at the position of the rf cavities. Thus, it is natural to put $\phi_0 = \omega_{\rm rf}(t_0)t_0$. Also, we put $\Delta\phi = \omega_{\rm rf}(t_0)\tau$ to the first order in τ. Here, $\omega_{\rm rf}(t)$ is the rf angular frequency. The conventional definition $\phi = \int^t \omega_{\rm rf}(t)dt$ is valid only for the traveling-wave approximation. Using $\Delta\phi$ in Eq. (21), combining the two as before, and neglecting the second and higher order terms in the adiabatically changing variables, we obtain after several steps

$$\frac{d}{d\theta}\left(\frac{\beta_0^2 E_0}{h\eta\omega_0}\frac{d\Delta\phi}{d\theta}\right) = \frac{eV}{2\pi\omega_0}\{\sin(\phi_0 + \Delta\phi) - \sin\phi_0\}, \tag{23}$$

where h is the harmonic number. If we put $\alpha = 1$ (pure bending field) and $h = 1$, this equation reduces to that by McMillan. Also, if we approximate $d\theta = \omega_0 dt$, Eq. (23) reduces to the equation given by Courant and Snyder [4].

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