Direct Numerical Simulation of Downshift and Inverse Cascade for Water Wave Turbulence

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By means of direct numerical simulations (DNS) based on the integrodifferential Zakharov equation, we study the long-term evolution of nonlinear random water wave fields. For the first time, formation of powerlike Kolmogorov-type spectra corresponding to weak-turbulent inverse cascade is demonstrated by DNS, and the evolution in time of the resulting spectra is quantitatively investigated. The predictions of the statistical theory for water waves, both qualitative (formation of the direct and inverse cascades, self-similar behavior) and quantitative (the spectra exponents, specific shape of self-similar functions, the rate of time evolution) are found to be in good agreement with the DNS results, except for the initial part of the evolution, where the established statistical theory is not applicable yet and the evolution has a much faster time scale.

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Introduction.—Wind waves on the ocean surface, due to their importance for navigation and essential role in weather forecasting, represent the most studied example of *wave* (or *weak*) *turbulence*. The term is commonly used to designate random motion of continuous weakly nonlinear dispersive wave fields in various physical contexts, including solid state physics, plasma physics and geophysical hydrodynamics (see, e.g., Ref. [1]). The affinity of wave turbulence with the classical hydrodynamic turbulence is more than merely semantic. Its key feature, discovered by Zakharov and Filonenko [2], is the existence of the Kolmogorov-type cascades of energy and other integrals of motion [1]. On the other hand, in contrast to the classical turbulence, there is a well-established general formalism for treating weakly nonlinear wave fields that exploits smallness of nonlinearity and a number of subtle assumptions about quasi-Gaussianity of a statistically homogeneous wave field. This approach, pioneered by Peierls [3], leads to a closed equation for the second statistical momenta of the field which we will refer to as the *kinetic equation* (KE). Although the theory based upon the KE was used in many different applications (e.g., Ref. [1]), the basic question—to what extent the theory corresponds to actual behavior of physical systems remains open, and the range of its validity, has not been established. Thus, an independent corroboration, which would be able to establish the range of validity of the theory, is essential. This Letter aims at this gap.

The most obvious way to validate the theory, and to get understanding of the statistical evolution when the theory is not applicable, is the direct numerical simulation (DNS) of statistical ensembles of random wave fields. Although this task seems to be straightforward, there are fundamental difficulties in applying DNS to this problem. First, the hydrodynamic equations should be integrated for a very large number of modes over quite large time intervals, much larger than those required in simulations of classical turbulence. The necessary times are at least $O(\varepsilon^{-4})$ of characteristic wave periods for the media with prevailing quartet interaction, as is the case for water waves. Here ε, the parameter that characterizes nonlinearity, is small by the definition of wave turbulence. Second, the KE describes continuous wave fields, and is essentially based on the existence of a continuum of waves involved in both resonant and nonresonant interactions. In this context it is not *a priori* clear how to perform the unavoidable discretization of a continuous wave field. Third, although there exists a large variety of algorithms developed to simulate evolution of water waves, none of them has proved to be particularly well suited to the challenge. Nevertheless, many different groups employed some modifications of the existing algorithms, although with a limited success [4–8].

The most robust predictions of the KE for dispersive waves with prohibited triad interactions are: (i) the existence of Kolmogorov-type cascades: the "direct" energy cascade towards small scales, which in terms of energy frequency spectra $E(\omega)$ manifests itself as a powerlike spectrum (in the water wave context $E(\omega) \sim \omega^{-4}$), and the "inverse" wave action cascade towards large scales, which also gives rise to powerlike spectrum with a different exponent [for water waves, $E(\omega) \sim \omega^{-11/3}$]; (ii) the scaling of energy fluxes and evolution times as $\sim\varepsilon^6$ and $\sim\varepsilon^{-4}$ respectively; (iii) solutions of the KE tend to become self-similar at large times, for a wide range of generation and dissipation conditions [9].

Based upon coarse mesh simulations, a preliminary type observation of the existence of the direct cascade for water waves was reported in Ref. [4]. In Ref. [5] the very initial stage of wave field evolution was simulated, and an agreement in energy fluxes between the DNS and KE was found. Recently the field evolution was simulated over $O(10^3)$ characteristic wave periods [6–8]. These works are based on the amplitude expansions of the original primitive hydrodynamic equations for the free-surface flow and the subsequent integration of the resulting equations using spectral methods, employing efficient realizations of fast Fourier transform. It was shown that there is indeed a direct

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cascade resulting in the formation of stationary powerlike spectra that are in a reasonable agreement with the predictions of the KE. However, there were no attempts to perform a detailed comparison of the evolution processes obtained with the KE and the DNS. In particular, the evolution time scales were not discussed. We also note that the ω^{-4} energy spectrum, in itself, is not a clinching argument in support of the theory, since this spectrum could be obtained on the basis of a simple dimensional analysis without resorting to any theoretical model [10,11]. A more sophisticated dimensional analysis within the KE paradigm yields the exponents of both direct and inverse cascades [12]. The main counterintuitive feature of wave turbulence, formation of the inverse cascade, has not been confirmed so far by DNS.

The common origin of the difficulties lies in the fact that the commonly employed fast techniques require the discretization of a wave field to a regular grid of Fourier harmonics. Regular grids are known to bring undesirable artifacts in behavior of wave turbulence [13]. In our context, it is especially important that the number of both exact and approximate resonances in a regular grid of harmonics is very limited, a very large grid being necessary for a successful modelling of weak turbulence. Moreover, a regular grid is, in fact, an unnatural representation of a continuous wave field with waves of very different scales. Even if the grid is sufficiently dense for short waves to form a rich enough system of resonances in the high-frequency part of the spectrum, the lack of resonances in the low-frequency part is inevitable, unless very substantial computational resources are used, such resources being clearly beyond the current capabilities.

In this Letter, we study formation and evolution of the inverse cascade employing a new numerical approach to the DNS of statistical ensembles of water waves, based on the integrodifferential Zakharov equations. This approach is not restricted to regular grids of harmonics. Instead, the wave field is represented as an ensemble of a large $[O(10^3\text{–}10^4)]$ number of finite-size wave packets with random phases, linked by a dense grid of approximate resonances, which represent the interactions of individual harmonics within wave packets. We simulate the evolution of an initially sharp impulse in the wave field in the presence of small dissipation at large wave numbers, as well as the evolution of wave turbulence excited by external source in higher-frequency part of the spectrum. Formation of powerlike spectra corresponding to the inverse cascades is demonstrated for the first time by DNS. Time evolution is shown to be generally in accordance with the predictions based upon the KE, in particular, the numerical solutions have a well-pronounced self-similar behavior. The area of sharp discrepancy between the KE and DNS is the initial stage of evolution: as it was suggested in Ref. [14], wave fields initially evolve on "dynamic" $O(\varepsilon^{-2})$ rather than the $O(\varepsilon^{-4})$ "kinetic" time scale.

Theoretical background.—Governing equations for potential gravity waves on the surface of ideal incompressible fluid of infinite depth have the Hamiltonian form

$$\frac{\partial\zeta(\mathbf{x},t)}{\partial t} = \frac{\delta H}{\delta\psi(\mathbf{x},t)}, \qquad \frac{\partial\psi(\mathbf{x},t)}{\partial t} = -\frac{\delta H}{\delta\zeta(\mathbf{x},t)},$$

where δ denotes the operator of functional differentiation, and the Hamiltonian H is the total energy of the system,

$$H = \frac{1}{2}\int_{-\infty}^{\zeta}\left[(\nabla\varphi)^2 + \left(\frac{\partial\varphi}{\partial z}\right)^2\right]dz d\mathbf{x} + \frac{1}{2}g\int\zeta^2 d\mathbf{x}.$$

Here, integration with respect to $\mathbf{x}$ over the entire horizontal plane is implied; $z = \zeta(\mathbf{x},t)$ specifies the surface, and $\varphi(\mathbf{x},z,t)$ is the velocity potential, with $\psi(\mathbf{x},t) = \varphi(\mathbf{x},\zeta(\mathbf{x},t),t)$ being the potential at the surface.

Provided that wave slopes are $O(\varepsilon)$ small, the Hamiltonian in Fourier space can be expanded in powers of ε. For gravity waves, triplet resonant interactions are not permitted, and the appropriate canonical transformation allows one to get rid of the cubic terms in the Hamiltonian, leading to the Zakharov equation [15]

$$\mathrm{i}\,\frac{\partial b_0}{\partial t} = (\omega_0 + \mathrm{i}\gamma_0)b_0 + \int T_{0123}b_1^*b_2b_3\delta_{0+1-2-3}d\mathbf{k}_{123}, \tag{1}$$

Here, $b(\mathbf{k})$ is a canonical complex variable, $\omega(\mathbf{k}) = (gk)^{1/2}$ is the linear dispersion relation, $i\gamma(\mathbf{k})$ is the small imaginary correction to frequency due to forcing or dissipation, g is normalized to unity, $k = |\mathbf{k}|$, integration in (1) is performed over the entire $\mathbf{k}$ plane. The compact notation used designates the arguments by indices, e.g., $T_{0123} = T(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)$; $\delta_{0+1-2-3} = \delta(\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3)$; asterisk means complex conjugation; t is time. All the details of the lengthy procedure of derivation of (1), as well as the expression for the kernel T can be found in Ref. [15]. The canonical variable $b(\mathbf{k})$ is linked to the Fourier-transformed primitive physical variables $\varphi(\mathbf{k},t)$ and $\zeta(\mathbf{k},t)$ through an integral-power series [15]

$$b(\mathbf{k}) = \frac{1}{\sqrt{2}}\left\{\sqrt{\frac{\omega(\mathbf{k})}{k}}\zeta(\mathbf{k}) + \mathrm{i}\sqrt{\frac{k}{\omega(\mathbf{k})}}\varphi(\mathbf{k})\right\} + O(\varepsilon).$$

The statistical description of wave fields is usually sought in terms of correlation functions of $b(\mathbf{k},t)$. The classical derivation (see, e.g., Ref. [1]) uses (1) as the starting point and leads to the equation

$$\frac{\partial n_0}{\partial t} = 4\pi\int T_{0123}^2 f_{0123}\delta_{0+1-2-3} \times \delta(\omega_0+\omega_1-\omega_2-\omega_3)d\mathbf{k}_{123}, \tag{2}$$

where n_0 is the second-order correlator, $\langle b_0^*b_1\rangle = n_0\delta_{0-1}$, angular brackets mean ensemble averaging, and $f_{0123} = n_2n_3(n_0+n_1) - n_0n_1(n_2+n_3)$. Equation (2) describes the evolution of the wave statistical ensemble and in the context of wind waves is often referred to as the Hasselmann equation; we use the term KE to emphasize its universality and relevance for all dispersive wave systems with prohibited triads. For its verification, we will focus on comparing the DNS and self-similar solutions to (2) discussed in Ref. [9].

Numerical algorithm.—The most natural way to study validity of (2) by means of DNS is to simulate numerically (1) from which it is derived. A new algorithm was proposed and successfully applied in Refs. [14,16] to the study of the evolution of relatively simple wave systems. In order to use it for DNS, it is necessary to work out a procedure of discretization of a continuous wave field, representing it in terms of a large number of harmonics coupled by exactly and approximately resonant interactions. We note that exact resonances do not play any special role in the field evolution [14].

Meanwhile, for a successful modelling of wave turbulence, it is essential to represent the continuous nature of resonant interactions within a wave field. For this purpose, we build a grid consisting of a moderate number $[O(10^3\text{–}10^4)]$ of *wave packets*, which comprise a much larger number $[O(10^7\text{–}10^8)]$ of individual harmonics, coupled through exact and approximate resonant interactions. A wave packet has the amplitude of the envelope of harmonics it comprises and a randomly chosen initial phase. Then, Zakharov equations in the form (1) are written for all wave packets, with all resonant interactions of the individual harmonics taken into account. Employing the smoothness of interaction coefficients, these interactions, linking individual harmonics of wave packets, are included into the equations for their central harmonics, taking into account the relevant phase correction terms. Although this procedure does not yet have a mathematically rigorous justification, the results of the simulations were thoroughly verified to be independent of the parameters of the method. The resulting system of Eq. (1) is solved numerically with the fourth-order Runge-Kutta method. All computations were performed on single-processor workstations.

Results.—In our numerical experiments, we study the evolution of initially isotropic spectra. We have chosen a grid of circular shape with regular spacing between wave packets, $\mathbf{k} = \pm\Delta(m + 1/2)\mathbf{i} \pm \Delta(n + 1/2)\mathbf{j}$, where $\Delta = 30$, $m, n = 0, 1, \ldots,$ and $\mathbf{i}$, $\mathbf{j}$ are unit vectors in Fourier space, under the condition that $|\mathbf{k}| < 1000$. We note that this problem does not have a length scale.

Let us first consider a classical test case problem of decay of a free localized impulse. To this end we prescribe an initially isotropic spectrum centered around a high wave number $|\mathbf{k}_0| = 600$, so that initially all the energy is in the high wave number part of the spectrum. Small dissipation at high wave numbers only, $|\mathbf{k}| > 784$, is inserted into (1), with $\gamma = -2.5 \times 10^{-12} A^3$, where A is the average amplitude in the wave number band $729 < |\mathbf{k}| < 784$, adjacent to the dissipation domain. Time t is measured in periods of $|\mathbf{k}_0|$. Long-term evolution of the wave number spectrum, up to $t = 10^5$ periods, is shown in Fig. 1. Formation of the powerlike spectrum, corresponding to the flux of wave action to lower frequencies and very close to the theoretical power k^{-4}, is demonstrated.

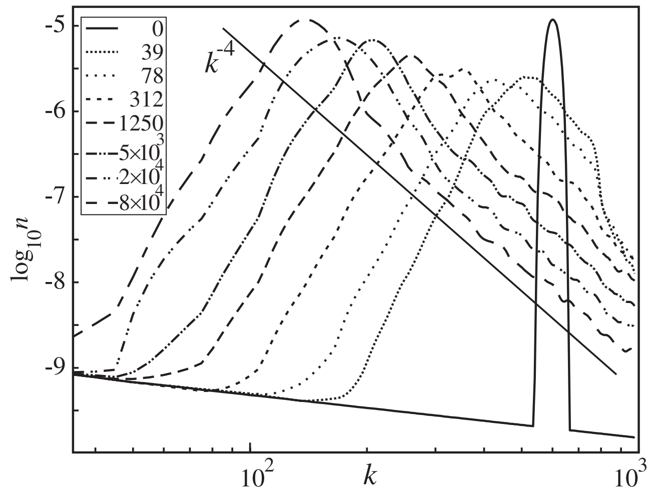

FIG. 1. Decay of an initial impulse without forcing. DNS wave action spectra for different times, measured in periods of the central harmonic of the impulse and shown in the box. Theoretical powerlike wave action spectrum k^{-4} is shown.

According to Ref. [9], the KE solutions for localized initial conditions without forcing tend to a self-similar asymptotics, $n = at^{4/11}U(\xi)$, where $\xi = \omega^2 t^{2/11}$. To demonstrate the self-similarity of the spectra shown in Fig. 1, function $U(\xi)$, extracted from the spectra at different times, is shown in Fig. 2. The self-similar function is very close to that obtained in Ref. [9], except for the somewhat less steep front. The initial evolution of the spectrum (at $t < 150$) does not have this self-similar behavior. A comparison with the evolution of more intense initial impulses shows that at relatively small times the time scale of the evolution is quadratic with respect to nonlinearity, not of fourth power as predicted by the KE. The existence of a substantial initial evolution occurring on

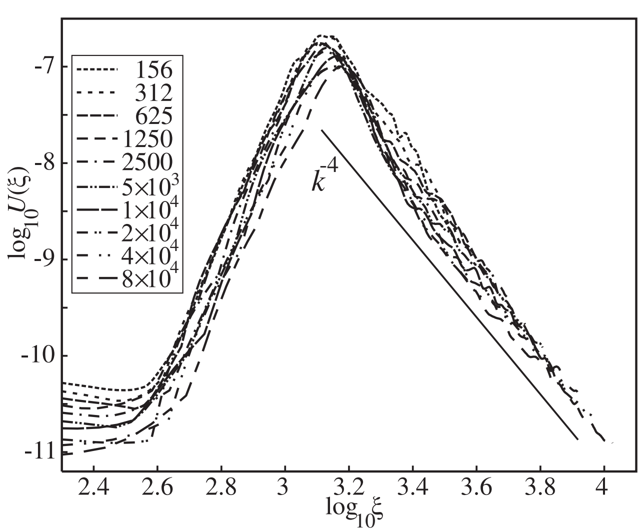

FIG. 2. Self-similar function $U(\xi)$, $\xi = \omega^2 t^{2/11}$, extracted from wave number spectra at different times, shown in the box. Theoretical powerlike spectrum k^{-4} is shown.

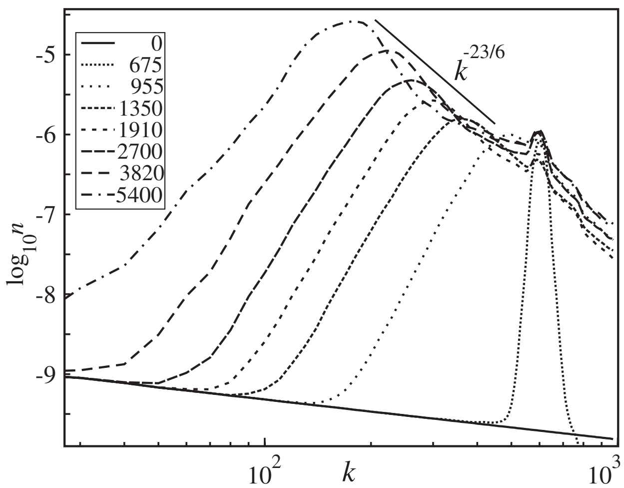

FIG. 3. Wave action spectra with forcing, obtained by DNS, for different times (shown in the box), measured in periods of the harmonic at the center of forcing. Theoretical powerlike spectrum $k^{-23/6}$ is shown by solid line.

a much faster time scale supports the earlier findings obtained for model problems [14].

As a more realistic example, consider another classical test case, that of forced turbulence, isotropically excited by external force at higher wave number part of the spectrum. We take the initial spectrum as a low-intensity white noise, and insert damping of small scales with $\gamma = -0.001$, for $|\mathbf{k}| > 784$ as before, and weak pumping, with $\gamma = 0.001$, for all $576 < |\mathbf{k}| < 625$. The long-term evolution of the spectrum, with averaging over 14 realizations, is shown in Fig. 3. Again, powerlike spectrum corresponding to inverse cascade is formed, and the evolution obtained with DNS is again self-similar, in accordance with the results of Ref. [9] for the KE. In Fig. 4, we show that the rate of downshift of the spectral peak frequency is close to the theoretical rate $t^{-3/11}$ [9]. The results were tested to be robust with respect of the choice of the grid, the number of harmonics, and other parameters of the numerical method.

Conclusions.—In this Letter, the new numerical approach has allowed us to trace with DNS the long-term evolution of weak turbulence of surface gravity waves. This, for the first time, has enabled us to check all the main features of wave turbulence predicted by the statistical theory: formation of the powerlike spectra corresponding to both the direct and inverse cascades, and the self-similar character of solutions. Thus, in the first approximation, we have verified the KE and justified its use in numerous applications. The only area of a major discrepancy identified so far is the initial [roughly $O(\varepsilon^{-3})$ wave periods] stage of field evolution: spectra evolve on faster $O(\varepsilon^{-2})$ time scale, rather than $O(\varepsilon^{-4})$ predicted by the KE, which is consistent with the recent findings on the role of approximately resonant interactions [14,17]. This observation has far reaching implications and, therefore, requires a

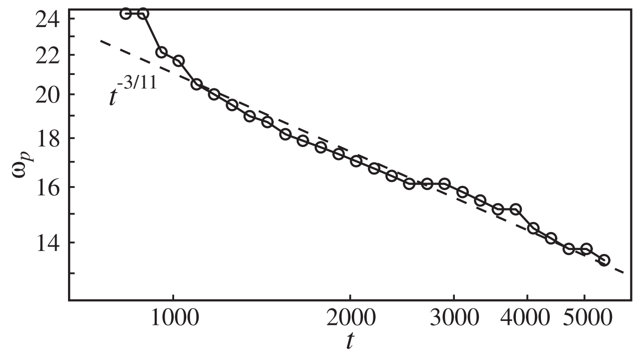

FIG. 4. Dependence of the peak frequency of the spectra ω_p on time. Theoretical downshift rate is shown by dashed line.

special study. A more extensive study is also needed to identify more subtle discrepancies and the underlying reasons. The results and conclusions reported in this Letter are of a general character and are applicable to any generic weakly nonlinear dispersive wave field with prohibited triad interactions.

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