Gravitational Instantons from Gauge Theory

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A gauge theory can be formulated on a noncommutative (NC) spacetime. This NC gauge theory has an equivalent dual description through the so-called Seiberg-Witten (SW) map in terms of an ordinary gauge theory on a commutative spacetime. We show that all NC $U(1)$ instantons of Nekrasov-Schwarz type are mapped to asymptotically locally Euclidean (ALE) gravitational instantons by the exact SW map and that the NC gauge theory of $U(1)$ instantons is equivalent to the theory of hyper-Kähler geometries. It implies the remarkable consequence that ALE gravitational instantons can emerge from local condensates of purely NC photons.

It is believed that spacetime must change its nature at distances comparable to the Planck scale. That is, spacetime at short distances is described by noncommutative (NC) geometry; where the spacetime coordinates do not commute, order is important. It may help understanding nonlocality at short distances since there are reasons to believe that any theory of quantum gravity will not be local in the conventional sense.

Consider a NC spacetime defined by

$$[y^\mu, y^\nu]_* = i\theta_{\mu\nu} \tag{1}$$

with a constant 4×4 matrix $\theta_{\mu\nu}$. Gauge theories can be constructed on this NC spacetime. For example, the action for NC $U(1)$ gauge theory in flat Euclidean $\mathbf{R}^4$ is given by

$$\hat{S}_{\text{NC}} = \frac{1}{4} \int d^4y \hat{F}_{\mu\nu} * \hat{F}^{\mu\nu}, \tag{2}$$

where noncommutative fields are defined by

$$\hat{F}_{\mu\nu} = \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu - i[\hat{A}_\mu, \hat{A}_\nu]_*. \tag{3}$$

It was shown in [1] that this NC $U(1)$ gauge theory admits nonsingular instanton solutions satisfying the NC self-duality equation,

$$\hat{F}_{\mu\nu}(y) = \pm \frac{1}{2} \varepsilon_{\mu\nu\alpha\beta} \hat{F}_{\alpha\beta}(y). \tag{4}$$

The NC gauge theory has an equivalent dual description through the so-called Seiberg-Witten (SW) map in terms of ordinary gauge theory on commutative spacetime [2]. The SW map is a map between gauge orbit spaces of commutative and NC gauge fields. It was shown in [3,4] that the commutative nonlinear electrodynamics equivalent to Eq. (2) is given by

$$S_C = \frac{1}{4} \int d^4x \sqrt{\det g} g^{\mu\alpha} g^{\beta\nu} F_{\mu\nu} F_{\alpha\beta}, \tag{5}$$

where we introduced an "effective metric" induced by the dynamical gauge fields such that

$$g_{\mu\nu} = \delta_{\mu\nu} + (F\theta)_{\mu\nu},$$
$$(g^{-1})^{\mu\nu} \equiv g^{\mu\nu} = \left(\frac{1}{1 + F\theta}\right)^{\mu\nu}. \tag{6}$$

The commutative action (5) can actually be derived from the NC action (2) using the exact SW maps in [4] (see [5] for the exact inverse SW map):

$$\hat{F}_{\mu\nu}(y) = \left(\frac{1}{1 + F\theta} F\right)_{\mu\nu}(x), \tag{7}$$

$$d^4y = d^4x \sqrt{\det(1 + F\theta)}(x), \tag{8}$$

where $x^\mu(y) \equiv y^\mu + \theta^{\mu\nu} \hat{A}_\nu(y)$. It was shown in [6] that the self-duality equation for the action S_C is given by

$$\mathbf{F}_{\mu\nu}(x) = \pm \frac{1}{2} \varepsilon_{\mu\nu\alpha\beta} \mathbf{F}_{\alpha\beta}(x), \tag{9}$$

where

$$\mathbf{F}_{\mu\nu}(x) \equiv (g^{-1}F)_{\mu\nu}(x). \tag{10}$$

The above equation is directly obtained by the exact SW map (7) from the NC self-duality Eq. (4). It was checked in [6] that the field configuration that satisfies the self-duality Eq. (9) also satisfies the equations of motion derived from the action (5).

A general strategy was suggested in [6] to solve the self-duality Eq. (9). For example, let us consider the anti-self-dual (ASD) case. Take a general ansatz for the ASD $\mathbf{F}_{\mu\nu}$ as follows:

$$\mathbf{F}_{\mu\nu}(x) = f^a(x) \bar{\eta}^a_{\mu\nu}, \tag{11}$$

where $\bar{\eta}^a$ are three 4×4 ASD 't Hooft matrices and f^a's are arbitrary functions. Then the Eq. (9) is automatically satisfied. Next, solve the field strength $F_{\mu\nu}$ in terms of $\mathbf{F}_{\mu\nu}$:

$$F_{\mu\nu}(x) = \left(\frac{1}{1 - \mathbf{F}\theta} \mathbf{F}\right)_{\mu\nu}(x), \tag{12}$$

and impose the Bianchi identity for $F_{\mu\nu}$,

$$\varepsilon_{\mu\nu\rho\sigma}\partial_\nu F_{\rho\sigma} = 0, \tag{13}$$

since the field strength $F_{\mu\nu}$ is given by a (locally) exact 2-form, i.e., $F = dA$. In the end one can get general differential equations governing $U(1)$ instantons [6].

Now let us restrict to the self-dual (SD) NC $\mathbf{R}^4$ properly normalized as $\theta_{\mu\nu} = \frac{1}{2}\eta^3_{\mu\nu}$ to consider the Nekrasov-Schwarz instantons [1]. The NC parameter θ can be easily recovered by a simple dimensional analysis by recalling that θ carries the dimension of (length)^2. It was shown [6] that the parameter θ, which settles the size of NC instantons, is related to the size of a minimal two-sphere known as a "bolt" in the gravitational instantons.

Substituting the ansatz (11) into Eq. (12), we get

$$F_{\mu\nu} = \frac{1}{1-\phi} f^a \bar{\eta}^a_{\mu\nu} - \frac{2\phi}{1-\phi}\eta^3_{\mu\nu}, \tag{14}$$

where $\phi \equiv \frac{1}{4}\sum_{a=1}^3 f^a(x)f^a(x)$. From Eq. (14), we also obtain

$$F^+_{\mu\nu} \equiv \frac{1}{2}\left(F_{\mu\nu} + \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}F_{\rho\sigma}\right) = \frac{1}{4}(F\tilde{F})\theta^+_{\mu\nu} \tag{15}$$

since

$$F\tilde{F} \equiv \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} = -\frac{16\phi}{1-\phi}. \tag{16}$$

We thus get

$$F_{24} = F_{13}, \qquad F_{14} = -F_{23}, \tag{17}$$

$$F_{12} + F_{34} = \frac{1}{4}F\tilde{F}. \tag{18}$$

Note that Eq. (15) is equivalent to the instanton equation in [2], although there it was derived perturbatively.

The metric for the $U(1)$ fields in Eq. (14) becomes symmetric, i.e., $g_{\mu\nu} = g_{\nu\mu}$ and its components are

$$g_{11} = g_{22} = 1 - \frac{1}{2}F_{12}, \qquad g_{33} = g_{44} = 1 - \frac{1}{2}F_{34},$$

$$g_{13} = g_{24} = -\frac{1}{2}F_{14}, \qquad g_{14} = -g_{23} = \frac{1}{2}F_{13}, \tag{19}$$

$$g_{12} = g_{34} = 0.$$

Equation (18) can be rewritten using the metric $g_{\mu\nu}$ as

$$g_{\mu\mu} = 4\sqrt{\det g_{\mu\nu}} \tag{20}$$

with $\sqrt{\det g_{\mu\nu}} = g_{11}g_{33} - (g_{13}^2 + g_{14}^2)$. We will show later that Eq. (20) reduces to the so-called complex Monge-Ampère equation, which is the Einstein field equation for a Kähler metric [7].

The Eq. (13) was solved in [6] for the single instanton case. It was found there that the effective metric (6) for the single $U(1)$ instanton is related to the Eguchi-Hanson (EH) metric [8], the simplest asymptotically locally Euclidean (ALE) space, and that the family of the EH space is parameterized by the instanton number. In this Letter we will show that the connection between NC $U(1)$ instantons

and hyper-Kähler geometries is more general. More precisely, we will see that the NC self-duality Eq. (4) is mapped by the SW map to gravitational instantons, defined by the following self-duality equation [9,10]

$$R_{abcd} = \pm\frac{1}{2}\varepsilon_{abef}R^{ef}_{cd}, \tag{21}$$

where R_{abcd} is a curvature tensor.

To proceed with the Kähler geometry, let us introduce the complex coordinates and the complex gauge fields

$$z_1 = x^2 + ix^1, \qquad z_2 = x^4 + ix^3, \tag{22}$$

$$A_{z_1} = A^2 - iA^1, \qquad A_{z_2} = A^4 - iA^3. \tag{23}$$

In terms of these variables, Eqs. (17) and (18) are written as

$$F_{z_1 z_2} = 0 = F_{\bar{z}_1 \bar{z}_2}, \tag{24}$$

$$F_{z_1 \bar{z}_1} + F_{z_2 \bar{z}_2} = -\frac{i}{4}F\tilde{F}, \tag{25}$$

where $F\tilde{F} = -4(F_{z_1\bar{z}_1}F_{z_2\bar{z}_2} + F_{z_1\bar{z}_2}F_{\bar{z}_1 z_2})$. Using the metric components in Eq. (19), one can easily see that the metric $g_{\mu\nu}$ is a Hermitian metric. That is,

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = g_{i\bar{j}}dz_i d\bar{z}_j, \qquad i,j = 1,2, \tag{26}$$

where

$$\begin{aligned}
g_{z_1\bar{z}_1} &= g_{11} = \frac{1}{1-\phi}\left(1 - \frac{f^3}{2}\right), \\
g_{z_2\bar{z}_2} &= g_{33} = \frac{1}{1-\phi}\left(1 + \frac{f^3}{2}\right), \\
g_{z_1\bar{z}_2} &= g_{13} - ig_{14} = \frac{1}{1-\phi}\frac{f^1 + if^2}{2}, \\
g_{\bar{z}_1 z_2} &= g_{13} + ig_{14} = \frac{1}{1-\phi}\frac{f^1 - if^2}{2}.
\end{aligned} \tag{27}$$

If we let

$$\Omega = \frac{i}{2}g_{i\bar{j}}dz_i \wedge d\bar{z}_j \tag{28}$$

be the Kähler form, then the Kähler condition is $d\Omega = 0$ [10], or, for all i, j, k,

$$\frac{\partial g_{i\bar{j}}}{\partial z^k} = \frac{\partial g_{k\bar{j}}}{\partial z^i}. \tag{29}$$

It is straightforward to check the pleasant property that the Kähler condition (29) is equivalent to the Bianchi identity (13) for the $U(1)$ field strength (14). Thus our metric $g_{i\bar{j}}$ is a Kähler metric and thus we can introduce a Kähler potential K defined by

$$g_{i\bar{j}} = \frac{\partial^2 K}{\partial z^i \partial \bar{z}^j}. \tag{30}$$

Now we will show that the Kähler potential K is related to the integrability condition of the self-duality Eq. (9). Locally, there is no difficulty in finding the general solution of Eq. (24):

$$A_{z_i} = 0, \qquad A_{\bar{z}_i} = 2i\partial_{\bar{z}_i}(K - \bar{z}_k z_k). \tag{31}$$

Then one can easily check that the real-valued smooth function K in Eq. (31) is equivalent to the Kähler potential in Eq. (30) (up to holomorphic diffeomorphisms).

As announced in Eq. (21), the metric $g_{\mu\nu}$ in Eq. (6) is related to gravitational instantons satisfying Eq. (21), whose metrics are necessarily Ricci flat. To see this, let us rewrite $g_{\mu\nu}$ as

$$g_{\mu\nu} = \frac{1}{2}(\delta_{\mu\nu} + \tilde{g}_{\mu\nu}). \tag{32}$$

Then, from Eq. (20), one can see that

$$\sqrt{\det \tilde{g}_{\mu\nu}} = 1. \tag{33}$$

Note that the metric $\tilde{g}_{\mu\nu}$ is also a Kähler metric:

$$\tilde{g}_{i\bar{j}} = \frac{\partial^2 \tilde{K}}{\partial z^i \partial \bar{z}^j}. \tag{34}$$

The relation $\det \tilde{g}_{\mu\nu} = (\det \tilde{g}_{i\bar{j}})^2$ definitely leads to

$$\det \tilde{g}_{i\bar{j}} = 1. \tag{35}$$

Therefore the metric $\tilde{g}_{\mu\nu}$ is both Ricci-flat and Kähler, which is the case of gravitational instantons [7]. For example, if one assumes that $\tilde{K}$ in Eq. (34) is a function solely of $r^2 = |z_1|^2 + |z_2|^2$, Eq. (35) can be integrated to give [11]

$$\tilde{K} = \sqrt{r^4 + t^4} + t^2 \log\frac{r^2}{\sqrt{r^4 + t^4} + t^2}. \tag{36}$$

This leads precisely to the EH metric of [6]. The instanton Eq. (15) is thus equivalent to the Einstein field equation for Kähler metrics.

Although it is obvious that the metric $\tilde{g}_{\mu\nu}$ in Eq. (32) is hyper Kähler [12], since in four dimensions the hyper-Kähler condition is equivalent to Ricci-flat Kähler, we would like to show more directly the hyper-Kähler structure. First we introduce a dual basis of 1-forms defined by $\sigma_\mu = \alpha_{\mu\nu}dx^\nu$, where, $\alpha_{\mu\nu} = \sqrt{\gamma g_{\mu\nu}}$ for $\mu = \nu$, $g_{\mu\nu}/2\sqrt{\gamma g_{11}}$ for $\mu \neq \nu$ and $\mu = 1, 2$, and $g_{\mu\nu}/2\sqrt{\gamma g_{33}}$ for $\mu \neq \nu$ and $\mu = 3, 4$, and

$$\gamma = \frac{1}{2}\left(1 \pm \frac{(\det g_{\mu\nu})^{1/4}}{\sqrt{g_{11}g_{33}}}\right).$$

Using the metric components in Eq. (19), one can show that the metric can be written as

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = \sigma_\mu \otimes \sigma_\mu \tag{37}$$

and

$$\sigma_1 \wedge \sigma_2 \wedge \sigma_3 \wedge \sigma_4 = \sqrt{\det g_{\mu\nu}}d^4x. \tag{38}$$

Let us introduce a SD local triple of 2-forms defined by

$$\omega^a = \frac{1}{2}\eta^a_{\mu\nu}\sigma^\mu \wedge \sigma^\nu, \tag{39}$$

where η^a are three 4×4 SD 't Hooft matrices. The explicit forms of ω^a are given by

$$\omega^1 = (\det g_{\mu\nu})^{1/4}(dx^1 \wedge dx^4 + dx^2 \wedge dx^3), \tag{40}$$

$$\omega^2 = (\det g_{\mu\nu})^{1/4}(dx^2 \wedge dx^4 + dx^3 \wedge dx^1), \tag{41}$$

$$\begin{aligned}\omega^3 &= g_{11}dx^1 \wedge dx^2 + g_{33}dx^3 \wedge dx^4 \\ &+ g_{13}(dx^1 \wedge dx^4 + dx^3 \wedge dx^2) \\ &+ g_{14}(dx^3 \wedge dx^1 + dx^4 \wedge dx^2).\end{aligned} \tag{42}$$

One can easily check that $d\omega^3 = 0$ if and only if the Bianchi identity (13) is satisfied. Since $\omega^3 = -\Omega$ in Eq. (28), this result indeed reproduces the Kähler condition (29). Thus the metric $g_{\mu\nu}$ is Kähler, as was shown before, but not hyper Kähler, since $d\omega^1 = d\omega^2 = 0$ requires $\det g_{\mu\nu} = $ constant.

However, if we consider the triple of Kähler forms of the metric $\tilde{g}_{\mu\nu}$ as follows,

$$\tilde{\omega}^a = \frac{1}{2}\eta^a_{\mu\nu}\tilde{\sigma}^\mu \wedge \tilde{\sigma}^\nu, \tag{43}$$

where $\tilde{\sigma}^\mu$ are defined in the same way as the σ^μs, but with $g_{\mu\nu}$ replaced by $\tilde{g}_{\mu\nu}$, we immediately get

$$d\tilde{\omega}^a = 0, \qquad \forall\ a, \tag{44}$$

since $\det \tilde{g}_{\mu\nu} = 1$. Thus the metric $\tilde{g}_{\mu\nu}$ should be a hyper-Kähler metric [12]. Therefore the hyper-Kähler condition has one-to-one correspondence with the self-duality equation of NC $U(1)$ instantons through the SW map.

Using this hyper-Kähler structure, we can easily understand the asymptotically locally flat (ALF) such as the Taub-NUT metric [9] as well as ALE instantons [8,13], which are a general class of self-dual, Ricci-flat metrics with a triholomorphic $U(1)$ symmetry. The metric is given by

$$ds^2 = U^{-1}(dx^4 + \vec{a} \cdot d\vec{x})^2 + U d\vec{x} \cdot d\vec{x}, \tag{45}$$

where x^4 parameterizes circles and $\vec{x} \in \mathbf{R}^3$. Since the above mentioned triholomorphic $U(1)$ symmetry is generated by the Killing field $\partial/\partial x^4$, the $U(1)$ invariant function $U = U(\vec{x})$ does not depend on x^4, and has to satisfy the condition

$$\vec{\nabla}U = \vec{\nabla} \times \vec{a}. \tag{46}$$

It turns out that the condition (46) is equivalent to the hyper-Kähler condition (44). To see this, let us introduce a 1-form basis as

$$\begin{aligned}\tilde{\sigma}^i &= \sqrt{U}dx^i, \qquad (i = 1, 2, 3), \\ \tilde{\sigma}^4 &= \frac{1}{\sqrt{U}}(dx^4 + \vec{a} \cdot d\vec{x}),\end{aligned} \tag{47}$$

where the metric reads as

$$ds^2 = \tilde{\sigma}_\mu \otimes \tilde{\sigma}_\mu. \tag{48}$$

It is then easy to see that Eq. (46) is equivalent to the hyper-Kähler condition (44) for the Kähler forms (43) given by the basis (47).

Thus the ALE and ALF spaces are all hyper-Kähler manifolds. But they have very different asymptotic behaviors of curvature tensors: The ALE spaces fall like $1/r^6$ while the ALF spaces fall like $1/r^3$ [10]. This suggests that the ALE metrics describe gravitational ''dipoles'' [8] while the ALF metrics describe monopoles (regarded as gravitational dyons) [9]. So ALF instantons may have a rather similar realization in terms of NC $U(1)$ instantons on $\mathbf{R}^3 \times S^1$: NC monopoles [14]. It seems to be in harmony with the fact that the ALF metrics approach a flat metric in the asymptotic $\mathbf{R}^3$ while the ALE metrics are asymptotically (locally) flat in $\mathbf{R}^4$.

In conclusion, we showed that NC instantons satisfying the self-duality Eq. (4) are all mapped through the SW map to gravitational instantons, precisely hyper-Kähler metrics, satisfying the self-duality Eq. (21). Our result implies that a nontrivial curved spacetime can emerge from local condensates of purely NC photons without any graviton. The manifestation of the emergent gravity from NC gauge theory is *not* in conflict with the Weinberg-Witten theorem [15], stating that an interacting graviton cannot emerge from an ordinary quantum field theory in the same dimensions, since NC field theories are neither local nor Lorentz invariant. But, recently, it was found [16] that NC field theory is invariant under the twisted Poincaré symmetry where the action of generators is now defined by the twisted coproduct in the deformed Hopf algebras. Although the energy-momentum tensor, being nonlinear in fields, is in NC field theories a twisted Poincaré invariant, the Weinberg-Witten theorem is still a consequence of the strict Lorentz invariance.

The connection between NC $U(1)$ instantons and gravitational instantons addresses several interesting issues. (These two seem to share at least two underlying mathematical structures: the twistor space and the holomorphic vector bundle, which may be important to understand why there exists such connection.) Using the connection, the problem of gravitational instantons can be addressed from the viewpoint of NC $U(1)$ instantons and vice versa. For example, the generalized positive action conjecture [17], gravitational anomalies and their index theorem [18], and the hyper-Kähler quotient construction of ALE spaces [19].

More extended work including the generalization of the emergent gravity from NC gauge theories beyond the self-dual sector will be reported elsewhere.

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