Hadronic Spectra and Light-Front Wave Functions in Holographic QCD

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(Received 28 February 2006; published 22 May 2006)

We show how the string amplitude $\Phi(z)$ defined on the fifth dimension in AdS_5 space can be precisely mapped to the light-front wave functions of hadrons in physical space-time. We find an exact correspondence between the holographic variable z and an impact variable ζ, which represents the measure of transverse separation of the constituents within the hadrons. In addition, we derive effective four dimensional Schrödinger equations for the bound states of massless quarks and gluons which exactly reproduce the anti–de Sitter conformal field theory results and give a realistic description of the light-quark meson and baryon spectrum as well as the form factors for spacelike Q^2. Only one parameter which sets the mass scale, Λ_{QCD}, is introduced.

DOI: 10.1103/PhysRevLett.96.201601 PACS numbers: 11.15.Tk, 11.25.Tq, 12.38.Aw, 12.40.Yx

The correspondence [1] between 10-dimensional string theory defined on $AdS_5 \times S^5$ and conformal Yang-Mills gauge theories in physical $3 + 1$ space-time has led to important insights into the properties of conformal theories at strong coupling. QCD is nearly conformal in the ultraviolet region. It is also a confining gauge theory in the infrared with a mass gap characterized by a scale Λ_{QCD} and a well-defined spectrum of color-singlet hadronic states. Although QCD is not conformal, many aspects of the theory, such as the dimensional scaling of exclusive amplitudes [2], follow if the QCD coupling has an infrared fixed point, allowing one to take conformal symmetry as an initial approximation.

The essential principle which leads to anti–de Sitter conformal field theory (AdS-CFT) duality is the fact that the group SO(2,4) of Lorentz and conformal transformations has a mathematical representation on AdS_5: the isomorphism of the group SO(2,4) of conformal QCD in the limit of massless quarks and vanishing β function with the isometries of AdS space, $x^\mu \to \lambda x^\mu$, $z \to \lambda z$, maps scale transformations into the holographic coordinate z, the extension of the hadron wave function into the fifth dimension. Different values of z determine the scale of the invariant separation between quarks. In particular, the $z \to 0$ boundary corresponds to the $Q \to \infty$, zero separation limit. As shown by Polchinski and Strassler [3], the resulting hadronic theory has the hard behavior and dimensional counting rules [2] expected from a conformal approximation to QCD, rather than the soft behavior characteristic of string theory.

Color confinement implies that there is a maximum separation of quarks and a maximum value of z. The cutoff at $z_0 = 1/\Lambda_{QCD}$ breaks conformal invariance and allows the introduction of the QCD scale. In fact, this holographic model gives a realistic description of the light-quark meson and baryon spectrum [4], including orbital excitations, as well as the meson and baryon form factors for spacelike Q^2. Remarkably, only one parameter, Λ_{QCD}, enters the predictions. Essential features of QCD, its near-conformal behavior at short physical distances, plus color confinement at large interquark separation, are incorporated in the model. This approach, known as holographic QCD, has been successful in obtaining general properties of the low-lying hadron spectra, chiral symmetry breaking, and hadron couplings [4,5].

The light-front wave functions $\psi^{S_z}_{n/h}(x_i, \mathbf{k}_{\perp i}, \lambda_i)$ of a hadron h encode all of its bound-state quark and gluon properties, including its momentum, spin, and flavor correlations, in the form of universal process- and frame-independent amplitudes. In this Letter we show how the string amplitude $\Phi(z)$ defined on the fifth dimension in AdS_5 space can be precisely mapped to the boost-invariant light-front wave functions of the hadrons. The resulting nonperturbative light-front wave functions and distributions allow the calculation of many observables including structure functions, distribution amplitudes, form factors, deeply virtual Compton scattering and decay constants. For example, the scale dependence of the string modes as determined from its twist dimension as one approaches the $z \to 0$ boundary determines the power-law behavior of the hadronic wave function at short distances, thus providing a precise counting rule for each Fock component with any number of quarks and gluons and any internal orbital angular momentum [6]. The prediction from AdS/CFT matches the behavior of the perturbative QCD results at small impact separation and at $x_i \to 1$ [7].

More generally, we show that there is an exact correspondence between the fifth dimensional variable z and a weighted impact separation variable ζ in physical space-time for each n-parton Fock state. Thus the coordinate z can be directly interpreted as a measure of the transverse separation of the constituents defined by the boost-invariant light-front wave function (LFWF) of the hadronic Fock state. In addition, we derive effective radial Schrödinger equations for the bound states of massless quarks and gluons where the effective potential is dictated by conformal symmetry and the constraint that the twist dimension of each hadron, including its orbital angular momentum, is reproduced at short distances. These effective equations for meson, baryons, and glueballs are completely equivalent to the AdS results.

The light-front Fock expansion of any hadronic system is constructed by quantizing quantum chromodynamics (QCD) at fixed light-cone time $x^+ = x^0 + x^3$ and forming the invariant light-cone Hamiltonian $H_{\rm LC}$: $H_{\rm LC} = P^2 = P^+P^- - \mathbf{P}_\perp^2$, with $P = (P^+, P^-, \mathbf{P}_\perp)$ [8,9]. In principle, solving the $H_{\rm LC}$ eigenvalue problem gives the entire mass spectrum of the color-singlet hadron states in QCD, together with their respective light-front wave functions. A hadronic state satisfies $H_{\rm LC}|\psi_h\rangle = M^2|\psi_h\rangle$, where $|\psi_h\rangle$ is an expansion in multiparticle Fock eigenstates $\{|n\rangle\}$ of the free light-front Hamiltonian: $|\psi_h\rangle = \sum_n \psi_{n/h}|\psi_h\rangle$. The solutions are independent of P^+ and $\mathbf{P}_\perp$. Thus, given the Fock projections $\psi^{S_z}_{n/h}(x_i, \mathbf{k}_{\perp i}, \lambda_i) = \langle n; x_i, \mathbf{k}_{\perp i}, \lambda_i|\psi_h(P^+, \mathbf{P}_\perp, S_z)\rangle$, the wave function of a hadron is determined in any frame [10]. The light-cone momentum fractions $x_i = k_i^+/P^+$ and $\mathbf{k}_{\perp i}$ represent the relative momentum coordinates of constituent i in Fock state n, and λ_i the helicity along the $\mathbf{z}$ axis. The physical momentum coordinates are k_i^+ and $\mathbf{p}_{\perp i} = x_i\mathbf{P}_\perp + \mathbf{k}_{\perp i}$. Here $\sum_{i=1}^n x_i = 1$ and $\sum_{i=1}^n \mathbf{k}_{\perp i} = 0$.

It is useful to define transverse position coordinates $x_i\mathbf{r}_{\perp i} = x_i\mathbf{R}_\perp + \mathbf{b}_{\perp i}$ so that $\sum_{i=1}^n \mathbf{b}_{\perp i} = 0$ and $\sum_{i=1}^n x_i\mathbf{r}_{\perp i} = \mathbf{R}_\perp$. The internal coordinates $\mathbf{b}_{\perp i}$ are conjugate to the relative coordinates $\mathbf{k}_{\perp i}$. The LFWF $\psi_n(x_j, \mathbf{k}_{\perp j})$ can be expanded in terms of the $n-1$ independent coordinates $\mathbf{b}_{\perp j}$, $j = 1, 2, \ldots, n-1$

$$\psi_n(x_j, \mathbf{k}_{\perp j}) = (4\pi)^{(n-1)/2} \prod_{j=1}^{n-1} \int d^2\mathbf{b}_{\perp j} \times \exp\left(i\sum_{j=1}^{n-1} \mathbf{b}_{\perp j}\cdot\mathbf{k}_{\perp j}\right)\tilde{\psi}_n(x_j, \mathbf{b}_{\perp j}). \tag{1}$$

The normalization is defined by

$$\sum_n \prod_{j=1}^{n-1} \int dx_j d^2\mathbf{b}_{\perp j}|\tilde{\psi}_n(x_j, \mathbf{b}_{\perp j})|^2 = 1. \tag{2}$$

The electromagnetic current $J^\mu(0)$ is represented in the interaction picture as a bilinear product of free fields, so that it has an elementary coupling to the charged constituent fields [11]. The Drell-Yan-West result for the form factor of a meson in terms of the transverse variables $\mathbf{b}_{\perp i}$ has the convenient form:

$$F(q^2) = \sum_n \prod_{j=1}^{n-1} \int dx_j d^2\mathbf{b}_{\perp j} \times \exp\left(i\mathbf{q}_\perp \cdot \sum_{j=1}^{n-1} x_j\mathbf{b}_{\perp j}\right)|\tilde{\psi}_n(x_j, \mathbf{b}_{\perp j})|^2, \tag{3}$$

corresponding to a change of transverse momentum $x_j\mathbf{q}_\perp$ for each of the $n-1$ spectators. The formula is exact if the sum is over all Fock states n. We use the standard light-cone frame where $q = (0, -q^2/P^+, \mathbf{q}_\perp)$ and $P = (P^+, M^2/P^+, \mathbf{0}_\perp)$. The momentum transferred by the photon to the system is $q^2 = -2P\cdot q = -\mathbf{q}_\perp^2$.

The form factor can be related to an effective single particle transverse density [12]

$$F(q^2) = \int_0^1 dx \int d^2\vec{\eta}_\perp e^{i\vec{\eta}_\perp\cdot\vec{q}_\perp}\tilde{\rho}(x, \vec{\eta}_\perp). \tag{4}$$

From (3) we find

$$\tilde{\rho}(x, \vec{\eta}_\perp) = \sum_n \prod_{j=1}^{n-1} \int dx_j d^2\mathbf{b}_{\perp j}\delta\left(1 - x - \sum_{j=1}^{n-1} x_j\right) \times \delta^{(2)}\left(\sum_{j=1}^{n-1} x_j\mathbf{b}_{\perp j} - \vec{\eta}_\perp\right)|\tilde{\psi}_n(x_j, \mathbf{b}_{\perp j})|^2, \tag{5}$$

where the integration is over the coordinates of the $n-1$ spectator partons, and $x = x_n$ is the coordinate of the active charged quark. We can identify $\vec{\eta}_\perp = \sum_{j=1}^{n-1} x_j\mathbf{b}_{\perp j}$. This is the x-weighted transverse position coordinate of the $n-1$ spectators. The procedure is valid for any n and thus the results can be summed over n to obtain an exact representation.

We now derive the corresponding expression for the form factor in AdS. The derivation can be extended to vector mesons and baryons. A nonconformal metric dual to a confining gauge theory is written as [3]

$$ds^2 = \frac{R^2}{z^2}e^{2A(z)}(\eta_{\mu\nu}dx^\mu dx^\nu - dz^2), \tag{6}$$

where $A(z) \to 0$ as $z \to 0$, and R is the AdS radius. In the "hard wall" approximation [3] the nonconformal factor $e^{2A(z)}$ is a step function: $e^{2A(z)} = \theta(z \le \Lambda_{\rm QCD}^{-1})$.

The hadronic matrix element for the electromagnetic current in the warped metric (6) has the form [13]

$$ig_5 \int d^4x dz\sqrt{g}A^\ell(x, z)\Phi^*_{P'}(x, z)\overleftrightarrow{\partial}_\ell\Phi_P(x, z). \tag{7}$$

We take an electromagnetic probe polarized along Minkowski coordinates, $A_\mu = \epsilon_\mu e^{-iQ\cdot x}J(Q, z)$, $A_z = 0$, where the function $J(Q, z)$ has the value 1 at zero momentum transfer, and as boundary limit the external current $A_\mu(x, z \to 0) = \epsilon_\mu e^{-iQ\cdot x}$. Thus $J(Q = 0, z) = J(Q, z = 0)$, since we are normalizing the bulk solutions to the total charge operator. The solution to the AdS wave equation, subject to boundary conditions at $Q = 0$ and $z \to 0$, is [13]

$$J(Q, z) = zQK_1(zQ). \tag{8}$$

The hadronic string modes are plane waves along the Poincaré coordinates with four-momentum P^μ and invariant mass $P_\mu P^\mu = \mathcal{M}^2$: $\Phi(x, z) = e^{-iP\cdot x}f(z)$. Substituting in (7) we find

$$F(Q^2) = R^3 \int_0^\infty \frac{dz}{z^3} e^{3A(z)} \Phi_{P'}(z) J(Q, z) \Phi_P(z). \tag{9}$$

The form factor in AdS is thus the overlap of the normalizable modes dual to the incoming and outgoing hadrons Φ_P and $\Phi_{P'}$ with the non-normalizable mode $J(Q, z)$ dual to the external source [13].

It is useful to integrate (4) over angles to obtain

$$F(q^2) = 2\pi \int_0^1 dx \frac{(1-x)}{x} \int \zeta d\zeta J_0\left(\zeta q \sqrt{\frac{1-x}{x}}\right) \tilde{\rho}(x, \zeta), \tag{10}$$

where we have introduced the variable

$$\zeta = \sqrt{\frac{x}{1-x}} \left| \sum_{j=1}^{n-1} x_j \mathbf{b}_{\perp j} \right|, \tag{11}$$

representing the x-weighted transverse impact coordinate of the spectator system. We also note the identity

$$\int_0^1 dx J_0\left(\zeta Q \sqrt{\frac{1-x}{x}}\right) = \zeta Q K_1(\zeta Q), \tag{12}$$

which is precisely the solution $J(Q, \zeta)$ for the electromagnetic potential in AdS (8). We can now see the equivalence between the LF and AdS results for the hadronic form factors. Comparing (10) with the expression for the form factor in AdS space (9), we can identify the spectator density function appearing in the light-front formalism with the corresponding AdS density

$$\tilde{\rho}(x, \zeta) = \frac{R^3}{2\pi} \frac{x}{1-x} e^{3A(\zeta)} \frac{|\Phi(\zeta)|^2}{\zeta^4}. \tag{13}$$

Equation (13) gives a precise relation between string modes $\Phi(\zeta)$ in AdS_5 and the QCD transverse charge density $\tilde{\rho}(x, \zeta)$. The variable ζ, $0 \leq \zeta \leq \Lambda_{\mathrm{QCD}}^{-1}$, represents the invariant separation between pointlike constituents, and it is also the holographic variable z characterizing the string scale in AdS; i.e., we can identify $\zeta = z$. For example, for two partons $\tilde{\rho}_{n=2}(x, \zeta) = |\psi(x, \zeta)|^2/(1-x)^2$, and a closed form solution for the two-constituent bound-state light-front wave function is found

$$|\tilde{\psi}(x, \zeta)|^2 = \frac{R^3}{2\pi} x(1-x) e^{3A(\zeta)} \frac{|\Phi(\zeta)|^2}{\zeta^4}. \tag{14}$$

In the case of two partons $\zeta^2 = \frac{x}{1-x} \vec{\eta}_\perp^2 = x(1-x)\mathbf{b}_\perp^2$.

In general, the short-distance behavior of a hadronic state is characterized by its twist (dimension minus spin) $\tau = \Delta - \sigma$, where σ is the sum over the constituent's spin $\sigma = \sum_{i=1}^n \sigma_i$. Twist is also equal to the number of partons $\tau = n$. Upon the substitution $\Delta \to n + L$, $\phi(z) = z^{-3/2}\Phi(z)$ in the AdS wave equations describing glueballs, mesons, or vector mesons [4], we find an effective Schrödinger equation as a function of the weighted impact variable ζ

$$\left[-\frac{d^2}{d\zeta^2} + V(\zeta) \right] \phi(\zeta) = \mathcal{M}^2 \phi(\zeta), \tag{15}$$

with the effective conformal potential [14]

$$V(\zeta) = -\frac{1 - 4L^2}{4\zeta^2}. \tag{16}$$

This new effective LF wave equation in physical space-time has stable solutions satisfying the Breitenlohner-Freedman bound [15]. The solution to (15) is

$$\phi(z) = z^{-3/2}\Phi(z) = C z^{1/2} J_L(z\mathcal{M}). \tag{17}$$

Its eigenvalues are determined by the boundary conditions at $\phi(z = 1/\Lambda_{\mathrm{QCD}}) = 0$ and are given in terms of the roots of the Bessel functions: $\mathcal{M}_{L,k} = \beta_{L,k}\Lambda_{\mathrm{QCD}}$. Normalized LFWFs $\tilde{\psi}_{L,k}$ follow from (14) [16]

$$\tilde{\psi}_{L,k}(x, \zeta) = B_{L,k}\sqrt{x(1-x)} J_L(\zeta \beta_{L,k} \Lambda_{\mathrm{QCD}}) \theta(z \leq \Lambda_{\mathrm{QCD}}^{-1}), \tag{18}$$

where $B_{L,k} = \Lambda_{\mathrm{QCD}}[(-1)^L \pi J_{1+L}(\beta_{L,k}) J_{1-L}(\beta_{L,k})]^{-1/2}$. The first eigenmodes are depicted in Fig. 1, and the masses of the light mesons in Fig. 2. The predictions for the lightest hadrons are improved relative to the results of Ref. [4] using the boundary conditions determined in terms of twist instead of conformal dimensions. The description of baryons is carried out along similar lines and will be presented elsewhere.

The holographic model is remarkably successful in organizing the hadron spectrum, although it underestimates the spin-orbit splittings of the $L = 1$ states. A better understanding of the relation between chiral symmetry breaking and confinement is required to describe successfully the pion. This would probably need a description of quark spin-flip mechanisms at the wall.

We have shown how the string amplitude $\Phi(z)$ defined on the fifth dimension in AdS_5 space can be precisely mapped to the frame-independent light-front wave functions of hadrons in physical space-time. This specific correspondence provides an exact holographic mapping at all energy scales between string modes in AdS and boundary

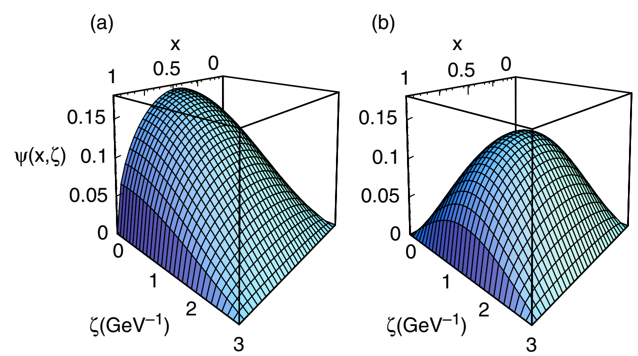

FIG. 1 (color online). Two-parton bound-state holographic LFWF $\tilde{\psi}(x, \zeta)$ for $\Lambda_{\mathrm{QCD}} = 0.32$ GeV: (a) ground state $L = 0$, $k = 1$, (b) first orbital excited state $L = 1$, $k = 1$.

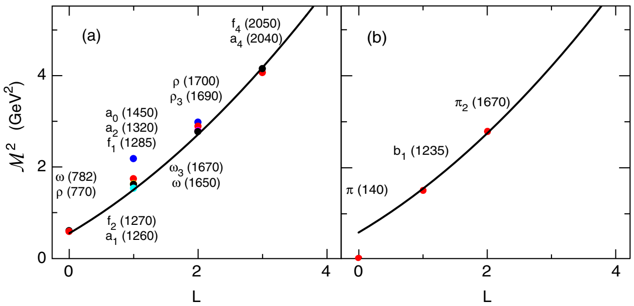

FIG. 2 (color online). Light meson orbital states for $\Lambda_{\rm QCD} = 0.32$ GeV: (a) vector mesons and (b) pseudoscalar mesons.

states with well-defined numbers of partons. Consequently, the AdS string mode $\Phi(z)$ can be regarded as the probability amplitude to find n partons at transverse impact separation $\zeta = z$. Its eigenmodes determine the hadronic mass spectrum. The degeneracy of hadron states depends on the flavor symmetry that is assumed, i.e., the number of massless quarks. There is no explicit dependence on N_C, and the QCD spectrum follows by matching twist dimensions to $SU(3)_C$ color-singlet hadronic states at the $z \to 0$ boundary.

The model can also be formulated in four dimensions without reference to AdS space [17]. To this end we have derived effective radial Schrödinger equations for the bound states of massless quarks and gluons with boundary conditions at zero separation distance determined by twist. These effective equations for meson, baryons, and glueballs exactly reproduce the AdS/CFT results. Since only one parameter is introduced, the agreement of the hadron spectrum with the observed pattern of physical states and the behavior of measured spacelike form factors is remarkable. When one includes the orbital dependence, the resulting set of wave functions are orthonormal and complete, giving a correct representation of current and charge matrix elements.

The phenomenological success of dimensional counting rules for exclusive processes can be understood if QCD resembles a strongly coupled conformal theory. The holographic model gives a mathematical realization of such theories. In some sense it is a covariant generalization of the MIT bag model, but it also incorporates the approximately conformal behavior of QCD at short physical distances. Our results suggest that basic features of QCD can be understood in terms of a higher dimensional dual gravity theory which holographically encodes multiparton boundary states into string modes and allows the computation of physical observables at strong coupling.

This research was supported by the Department of Energy Contract No. DE-AC02-76SF00515.

[1] J. M. Maldacena, Adv. Theor. Math. Phys. **2**, 231 (1998); S. S. Gubser, I. R. Klebanov, and A. M. Polyakov, Phys. Lett. B **428**, 105 (1998); E. Witten, Adv. Theor. Math. Phys. **2**, 253 (1998).
[2] S. J. Brodsky and G. R. Farrar, Phys. Rev. Lett. **31**, 1153 (1973); V. A. Matveev, R. M. Muradian, and A. N. Tavkhelidze, Lett. Nuovo Cimento **7**, 719 (1973).
[3] J. Polchinski and M. J. Strassler, Phys. Rev. Lett. **88**, 031601 (2002).
[4] G. F. de Teramond and S. J. Brodsky, Phys. Rev. Lett. **94**, 201601 (2005).
[5] C. Csaki, H. Ooguri, Y. Oz, and J. Terning, J. High Energy Phys. 01 (1999) 017; H. Boschi-Filho and N. R. F. Braga, J. High Energy Phys. 05 (2003) 009; M. Kruczenski, D. Mateos, R. C. Myers, and D. J. Winters, J. High Energy Phys. 07 (2003) 049; S. Hong, S. Yoon, and M. J. Strassler, J. High Energy Phys. 04 (2004) 046; S. Hong, S. Yoon, and M. J. Strassler, hep-ph/0501197; J. Babington, J. Erdmenger, N. J. Evans, Z. Guralnik, and I. Kirsch, Fortschr. Phys. **52**, 578 (2004); N. Evans and J. P. Shock, Phys. Rev. D **70**, 046002 (2004); M. Kruczenski, L. A. P. Zayas, J. Sonnenschein, and D. Vaman, J. High Energy Phys. 06 (2005) 046; T. Sakai and S. Sugimoto, Prog. Theor. Phys. **113**, 843 (2005); T. Sakai and S. Sugimoto, Prog. Theor. Phys. **114**, 1083 (2005); J. Erlich, E. Katz, D. T. Son, and M. A. Stephanov, Phys. Rev. Lett. **95**, 261602 (2005); L. Da Rold and A. Pomarol, Nucl. Phys. B **721**, 79 (2005); J. High Energy Phys. 01 (2006) 157; J. Hirn and V. Sanz, J. High Energy Phys. 12 (2005) 030; H. Boschi-Filho, N. R. F. Braga, and H. L. Carrion, Phys. Rev. D **73**, 047901 (2006); K. Ghoroku, N. Maru, M. Tachibana, and M. Yahiro, Phys. Lett. B **633**, 602 (2006); E. Katz, A. Lewandowski, and M. D. Schwartz, hep-ph/0510388; T. Hambye, B. Hassanain, J. March-Russell, and M. Schvellinger, hep-ph/0512089.
[6] S. J. Brodsky and G. F. de Teramond, Phys. Lett. B **582**, 211 (2004).
[7] X. Ji, J.-P. Ma, and F. Yuan, Phys. Rev. Lett. **90**, 241601 (2003).
[8] P. A. M. Dirac, Rev. Mod. Phys. **21**, 392 (1949).
[9] S. J. Brodsky, H. C. Pauli, and S. S. Pinsky, Phys. Rep. **301**, 299 (1998).
[10] G. P. Lepage and S. J. Brodsky, Phys. Rev. D **22**, 2157 (1980); S. J. Brodsky, J. R. Hiller, D. S. Hwang, and V. A. Karmanov, Phys. Rev. D **69**, 076001 (2004).
[11] S. D. Drell and T. M. Yan, Phys. Rev. Lett. **24**, 181 (1970); G. B. West, Phys. Rev. Lett. **24**, 1206 (1970); S. J. Brodsky and S. D. Drell, Phys. Rev. D **22**, 2236 (1980).
[12] D. E. Soper, Phys. Rev. D **15**, 1141 (1977).
[13] J. Polchinski and M. J. Strassler, J. High Energy Phys. 05 (2003) 012.
[14] The effective $q\bar{q}$ potential for heavy quarks at large N_C has been computed by S. J. Rey and J. T. Yee, Eur. Phys. J. C **22**, 379 (2001); J. Maldacena, Phys. Rev. Lett. **80**, 4859 (1998).
[15] P. Breitenlohner and D. Z. Freedman, Ann. Phys. (N.Y.) **144**, 249 (1982).
[16] LFWFs of this general form have been proposed in: A. V. Radyushkin, Acta Phys. Pol. B **26**, 2067 (1995).
[17] J. Erlich, G. D. Kribs, and I. Low, Phys. Rev. D **73**, 096001 (2006); D. T. Son and M. A. Stephanov, Phys. Rev. D **69**, 065020 (2004).