# Hidden Supersymmetry of Domain Walls and Cosmologies

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We show that all domain-wall solutions of gravity coupled to scalar fields for which the world-volume geometry is Minkowski or anti–de Sitter admit Killing spinors, and satisfy corresponding first-order equations involving a superpotential determined by the solution. By analytic continuation, all flat or closed Friedmann-Lemaître-Robertson-Walker cosmologies are shown to satisfy similar first-order equations arising from the existence of "pseudo Killing" spinors.

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Scalar fields arise naturally in many supergravity theories and the domain-wall solutions they allow are important for the holographic interpretation of renormalization group flow. They have also been invoked in the context of inflationary cosmology, and as a possible source of "dark energy." A general theoretical framework for these studies, in spacetime dimension $D = d + 1$, is provided by the Lagrangian density

$$\mathcal{L} = \sqrt{-\det g}[R - \tfrac{1}{2}|\partial\Phi|^2 - V(\Phi)] \qquad (1)$$

for metric $g$, with scalar curvature $R$, and scalar fields $\Phi$ taking values in some Riemannian target space, on which there is some potential energy function $V$. One purpose of this Letter is to exploit a close connection between domain-wall and cosmological solutions of the above model, but our initial focus will be domain walls because a domain-wall solution may be a supersymmetric solution of some supergravity theory for which (1) is a consistent truncation. In practice, this involves a determination of whether the solution admits Killing spinors, which are nonzero spinor fields annihilated by a covariant derivative operator constructed from the standard spin connection and a "superpotential," which determines the potential $V$ by a simple formula involving first derivatives.

There are many "flat" domain-wall solutions, with Minkowski $d$-dimensional geometry, that have long been known to be supersymmetric solutions of some supergravity theory. More recently, beginning with [1], supersymmetric *curved* domain-wall solutions have been found. Such results all depend on the specific superpotential of the supergravity theory under study, even though only the potential $V$ is relevant to the solution itself. Moreover, the superpotential is not uniquely defined by the potential, which means that there can be many supergravity theories with the *same* metric-scalar truncation; a solution that is non-supersymmetric for one supergravity theory could be a supersymmetric solution of another one. This state of affairs suggests a supergravity-independent definition of a "supersymmetric" domain-wall solution as one that admits a Killing spinor for *some* superpotential function that yields the given potential $V$ [2–6]; the superpotential then defines a "fake supergravity" [4]. This definition raises two related questions: which domain-wall solutions of a given model, with specified target space and potential $V$, are supersymmetric in the above sense, and which models admit such solutions?

These questions have been raised and partially answered in the recent literature [4–6]. Here we present an essentially complete answer to both questions for domain-wall solutions that are foliated (or "sliced") by $d$-dimensional de Sitter (dS), Minkowski, or anti–de Sitter (adS) spaces. Moreover, the answer is very simple, and model independent. *All* Minkowski and adS-sliced domain walls are supersymmetric for *any* model of the form (1), and the only dS-sliced domain walls that are supersymmetric are the dS foliation of $D$-dimensional Minkowski or adS space. There *are* some caveats, in particular, the result may be true only locally (in the many scalar case) or "piecewise" (if the superpotential turns out to be multivalued).

Although cosmologies cannot be supersymmetric (with the exception of anti–de Sitter space), first-order equations for flat cosmologies arise in the Hamilton-Jacobi formalism [7] and their similarity with Bogomolnyi-Prasad-Somerfeld (BPS) equations has been noted [8]. In addition, many previous works have obtained cosmological solutions of *particular* models by analytic continuation of domain-wall solutions. Here we establish a general result: for every domain-wall solution (of the type specified above) there is a corresponding Friedmann-Lemaître-Robertson-Walker (FLRW) cosmology, of the same model but with opposite sign potential. The generality of our result for domain walls then implies that *all* flat or closed FLRW cosmologies solve first-order "BPS-type" equations involving a superpotential determined by the solution, despite the fact that they are not supersymmetric. We will show that this result arises from the existence of "pseudo Killing" spinors.

As we wish to consider both domain walls and cosmologies, it is convenient to introduce a sign $\eta$ such that $\eta = 1$ for domain walls and $\eta = -1$ for cosmologies. Then, in either case, the metric for the solutions of interest here can be put in the form

$$ds_D^2 = \eta(fe^{\alpha\varphi})^2 dz^2 + e^{2\beta\varphi}\left[-\frac{\eta dr^2}{1+\eta kr^2} + r^2 d\Omega_\eta^2\right] \quad (2)$$

where $d\Omega_\eta^2$ is *either* (for $\eta = -1$) a unit radius $SO(d)$-invariant metric on the $(d-1)$ sphere, *or* (for $\eta = 1$) a unit radius $SO(1, d-1)$-invariant metric on the $(d-1)$-dimensional hyperboloid. For $\eta = -1$, $z$ is a time variable and the constant $z$ hypersurfaces are maximally symmetric spaces with inverse radius of curvature $k$ normalized to $k = -1, 0, 1$. For $\eta = 1$, $r$ is the time variable and the constant $z$ hypersurfaces are maximally symmetric $d$-dimensional spacetimes with inverse radius of curvature $k = -1, 0, 1$, and hence have adS, Minkowski, or dS geometry, respectively. In suitable coordinates for the metric $d\Omega_\eta^2$, the $D$-dimensional domain-wall and FLRW cosmology metrics are related by a double-analytic continuation.

We have allowed for an arbitrary function $f(z)$ in the ansatz (2) in order to maintain $z$-parametrization invariance, and we have introduced for later convenience the $D$-dependent constants

$$\alpha = (D-1)\beta, \qquad \beta = 1/\sqrt{2(D-1)(D-2)}. \quad (3)$$

The scalar fields must be taken to be functions only of $z$ in order to preserve the spacetime isometries. The field equations then reduce to equations for the variables $(\varphi, \Phi)$ that are equivalent to the Euler-Lagrange equations of the effective Lagrangian

$$L = \frac{\eta}{2} f^{-1}(\dot\varphi^2 - |\dot\Phi|^2) - fe^{2\alpha\varphi}\left(V - \frac{k}{2\beta^2}e^{-2\beta\varphi}\right), \quad (4)$$

where the overdot indicates differentiation with respect to $z$. Note that a change in the sign $\eta$ can be compensated by a change of sign of both $V$ and $k$, because the overall sign of $L$ has no effect on the equations of motion. It follows that for every domain-wall solution of a model with scalar potential $V$ there is a corresponding FLRW cosmology, with the opposite sign of $k$ if $k \neq 0$, for a model with the opposite sign of $V$.

For simplicity of presentation, we begin by supposing that there is only one field $\sigma$. Later we will show how our results generalize to the multiscalar case. We will also fix the $z$ reparametrization by the gauge choice $f = e^{-\alpha\varphi}$. In this gauge, and for a one-scalar model, the Euler-Lagrange equations of $L$ are equivalent to the equations

$$\ddot\varphi = -\alpha\dot\sigma^2 - (k\eta/\beta)e^{-2\beta\varphi}, \qquad \ddot\sigma = -\alpha\dot\varphi\,\dot\sigma + \eta V', \quad (5)$$

where the prime indicates differentiation with respect to $\sigma$, together with the constraint

$$\dot\varphi^2 - \dot\sigma^2 = -2\eta\left[V - \frac{k}{2\beta^2}e^{-2\beta\varphi}\right]. \quad (6)$$

If $V$ has an extremum that allows a solution with $\dot\sigma \equiv 0$ then the domain-wall or cosmological solution is actually a dS, Minkowski, or adS vacuum solution. So we shall assume that $\dot\sigma$ is not identically zero. In fact, we shall assume initially that $\dot\sigma$ is nowhere zero, returning subsequently to consider what happens when $\dot\sigma$ has isolated zeros. Given that $\dot\sigma \neq 0$, there is an inverse function $z(\sigma)$ that allows any function of $z$ to be considered as a function of $\sigma$. In particular, given any solution with $\eta k \leq 0$ for which $\dot\sigma \neq 0$, we may define a complex function

$$Z(\sigma) = \omega(\sigma)e^{i\theta(\sigma)} \quad (7)$$

by the formulas

$$\omega = \frac{1}{2\alpha}\sqrt{\dot\varphi^2 - \frac{k\eta}{\beta^2}e^{-2\beta\varphi}}, \quad (8)$$

$$\theta' = \pm\sqrt{-k\eta}\left(\frac{\alpha}{\beta}\right)\dot\sigma e^{-\beta\varphi}\left(\dot\varphi^2 - \frac{k\eta}{\beta^2}e^{-2\beta\varphi}\right)^{-1}. \quad (9)$$

Note that $\theta' = 0$ for $k = 0$ so in this case we may choose $\theta = 0$, and hence $Z = \omega$.

We claim that the function $Z(\sigma)$ constructed according to the above prescription satisfies

$$V = 2\eta[|Z'|^2 - \alpha^2|Z|^2] \quad (10)$$

as a consequence of the equations of motion, and further that the solution used to construct $Z(\sigma)$ satisfies

$$\dot\sigma = \pm 2|Z'|, \qquad \dot\varphi = \mp\frac{2\alpha}{|Z'|}\mathcal{R}e(\bar Z Z'),$$
$$-k\eta e^{2\beta\varphi} = (2\alpha\beta\,\mathrm{Im}(\bar Z Z')/|Z'|)^2. \quad (11)$$

In fact, these equations imply the second-order ones. Inserting (11) in (10) yields the constraint (6). Differentiating the first of (11) and using (10) and (11) yields the second of Eq. (5). Finally, the first of Eq. (5) follows directly from the second of Eq. (11) upon using the definitions of $\omega, \theta$ in (8) and (9). There is a consistency condition between the second and third of Eq. (11): $\dot\varphi$ computed from the third should agree with the second. This requires

$$\mathrm{Im}[\bar Z'(Z'' + \alpha\beta Z)] = 0. \quad (12)$$

Remarkably, this is an *identity* for $(\omega, \theta)$ defined by (8) and (9), so all $k\eta \leq 0$ solutions satisfy first-order equations, for either choice of the sign $\eta$.

As a concrete illustration of the above, consider the $D = 3$ model with $V = -\eta$, and the $k = -\eta$ solution [6]

$$e^{\varphi} = 1 + e^{\sqrt{2}z}, \qquad e^{-\sigma} = 1 + e^{-\sqrt{2}z}. \quad (13)$$

For $\eta = 1$ this yields an adS-sliced "separatrix wall" solution that interpolates between an $adS_2 \times R$ linear-dilaton vacuum (at $z = -\infty$) and the $adS_3$ vacuum (at $z = \infty$). For $\eta = -1$ it yields a $k = 1$ FLRW cosmology that interpolates between an Einstein static universe (supported by a constant $\sigma$ kinetic energy) in the far past and the $dS_3$ vacuum in the far future. Note that $\sigma < 0$ for this solution, so that $(1 - e^{\sigma})$ is positive. One finds that

$$\begin{aligned} \omega(\sigma) &= \sqrt{1 - e^{\sigma} + \frac{1}{2}e^{2\sigma}}, \\ \theta(\sigma) &= \arctan[e^{-\sigma}\sqrt{2(1 - e^{\sigma})}] \\ &\quad + \frac{1}{\sqrt{2}}\log\left(\frac{1 - \sqrt{1 - e^{\sigma}}}{1 + \sqrt{1 - e^{\sigma}}}\right) + \theta_0, \end{aligned} \tag{14}$$

for arbitrary, and irrelevant, constant $\theta_0$.

So far, we have considered domain walls and cosmologies on an equal footing, but we now restrict to the domain-wall case, $\eta = 1$. For this case, we claim that the first-order Eq. (11) are BPS equations that guarantee the existence of a Killing spinor field. It would be sufficient for our purposes to consider a complex superpotential modeled on minimal $D = 4$ supergravity but to make use of previous work on, or inspired by, minimal $D = 5$ supergravity we consider instead a real $Sp_1$-triplet superpotential $\mathbf{W}(\sigma)$ and a Killing spinor equation of the form [4,5,9]

$$(D_\mu - \alpha\beta\mathbf{W}\cdot\boldsymbol{\tau}\Gamma_\mu)\epsilon = 0, \qquad (\mu = 0, 1, \ldots, d), \tag{15}$$

where $D_\mu$ is the standard covariant derivative on spinors, and $\boldsymbol{\tau}$ is the triplet of Pauli matrices. In the context of minimal $D = 5$ supergravity, $\epsilon$ is an $Sp_1$-Majorana spinor and $\mathbf{W}$ is real. The reality of $\mathbf{W}$ is also required for the "gamma trace" of the Killing spinor equation to be a Dirac equation with a hermitian "mass" matrix, and this condition can (and should) be imposed as part of the definition of a "fake" Killing spinor. With this understood, we may allow $\epsilon$ in (15) to be a Dirac spinor in arbitrary spacetime dimension $D$.

For a solution of the assumed form, the Killing spinor Eq. (15) reduces to the equations

$$\begin{aligned} \partial_z\epsilon &= \alpha\beta\mathbf{W}\cdot\boldsymbol{\tau}\Gamma_{\underline{z}}\epsilon, \\ \hat{D}_m\epsilon &= e^{\beta\varphi}\hat{\Gamma}_m[(\beta/2)\dot{\varphi}\Gamma_{\underline{z}} + \alpha\beta\mathbf{W}\cdot\boldsymbol{\tau}]\epsilon, \end{aligned} \tag{16}$$

where $\Gamma_{\underline{z}}$ is a *constant* matrix that squares to the identity, and a hat indicates restriction to the (normalized) world-volume metric, so $\hat{\Gamma}_m$ are the world-volume Dirac matrices. The integrability conditions of these equations were discussed in Ref. [4] and we review this analysis here. The second of the Eq. (16) has the integrability condition

$$\dot{\varphi}^2 = 4\alpha^2|\mathbf{W}|^2 + (k/\beta^2)e^{-2\beta\varphi}. \tag{17}$$

We will now suppose that the potential $V$ is given in terms of the triplet superpotential by the relation

$$V = 2[|\mathbf{W}'|^2 - \alpha^2|\mathbf{W}|^2]. \tag{18}$$

At this point, the reader may guess how $\mathbf{W}$ is determined by the complex function $Z$ introduced earlier, but no guesswork is needed: the relation between the two will emerge from consistency requirements. Given the constraint (6) and the above form of the potential, (17) implies that

$$\dot{\sigma} = \pm 2|\mathbf{W}'|. \tag{19}$$

Differentiating (17) and using the equations of motion to eliminate $\ddot{\varphi}$, and then eliminating $V$ in favor of $\mathbf{W}$, we deduce the first-order equation

$$\dot{\varphi} = \mp 2\alpha(\mathbf{W}\cdot\mathbf{W}')/|\mathbf{W}'| \tag{20}$$

*and* the condition

$$|\mathbf{W}\times\mathbf{W}'|^2 = -k(D-2)^2e^{-2\beta\varphi}|\mathbf{W}'|^2. \tag{21}$$

It follows from this condition that a dS-sliced ($k = 1$) domain-wall can admit Killing spinors only if $\mathbf{W}$ is constant, which requires $\dot{\sigma} \equiv 0$ and implies that $V$ is a nonpositive constant; in this case the $D$-dimensional spacetime is a dS foliation of either Minkowski or adS space. Excluding these trivial cases, we conclude that a Killing spinor requires either $k = 0$ or $k = -1$, and that $k = 0$ requires $\mathbf{W}\times\mathbf{W}' = \mathbf{0}$, which implies that $\mathbf{W} = W\mathbf{n}$ for a singlet superpotential $W(\sigma)$ and a *fixed* 3-vector $\mathbf{n}$.

The Killing spinor Eq. (16) also have the joint integrability condition

$$(\dot{\sigma} + 2\mathbf{W}'\cdot\boldsymbol{\tau}\Gamma_{\underline{z}})\epsilon = 0, \tag{22}$$

which has the supergravity interpretation as the condition of vanishing supersymmetry variation of the superpartner to $\sigma$. This condition must be satisfied for all $z$; it is trivially satisfied if $\dot{\sigma} \equiv 0$ since (19) then implies that $\mathbf{W}$ is constant. Otherwise, it implies the projection

$$(1 \pm \Gamma)\epsilon = 0, \qquad \Gamma = (\mathbf{W}'/|\mathbf{W}'|)\cdot\boldsymbol{\tau}\Gamma_{\underline{z}}. \tag{23}$$

For $k = 0$ we have $\Gamma = (\mathbf{n}\cdot\boldsymbol{\tau})\Gamma_{\underline{z}}$, which is a constant traceless matrix that squares to the identity matrix, implying preservation of $1/2$ supersymmetry. Otherwise $\Gamma$ is not a constant matrix and differentiation with respect to $z$ of the projection condition yields the consistency condition

$$(\mathbf{W}'' + \alpha\beta\mathbf{W})\times\mathbf{W}' = \mathbf{0}. \tag{24}$$

This condition implies that $\mathbf{W}$ and all its derivatives are coplanar, so that $\mathbf{W} = X\mathbf{n} + Y\mathbf{m}$ for fixed orthonormal 3-vectors $\mathbf{n}$ and $\mathbf{m}$, and functions $X(\sigma)$, $Y(\sigma)$. The first-order equations (19) and (20) are then equivalent to the first-order equation (11) if we make the identification $Z = X + iY$. The integrability condition (17) is then equivalent to the equation (8) for $\omega(\sigma)$, and (21) is similarly equivalent to (11), which is itself equivalent to the equation (9) for $\theta(\sigma)$.

Thus, the complex function $Z$ appearing in (11) determines the triplet superpotential $\mathbf{W}$. In terms of $Z$, the consistency condition (24) is just (12) and, as already mentioned, this is satisfied identically. This means that all flat or adS-sliced domain-wall solutions of the one-scalar model for which $\dot{\sigma}$ does not vanish preserve $1/2$ supersymmetry for a superpotential that is determined by the solution, in the sense that they admit Killing spinors for this superpotential subject to (at most) a $1/2$ supersymmetry projection.

The condition of nonvanishing $\dot{\sigma}$ was needed because our construction of the superpotential assumed the existence of a function $z(\sigma)$ inverse to $\sigma(z)$. While this condition may be satisfied for many domain-wall solutions, others will typically have isolated values of $z$ for which $\dot{\sigma} = 0$ (for example, the "$\lambda$-perturbed Janus solutions" described in Ref. [6] all have one point at which $\dot{\sigma} = 0$). When this happens the inverse function $z(\sigma)$ will become multivalued, with different branches in intervals of $z$ on either side of a zero of $\dot{\sigma}(z)$. In other words, it will still be true that the domain-wall solution is supersymmetric for a superpotential determined by the solution, but this superpotential will be a multivalued function and more than one branch will be needed. Thus understood, our claim remains true piecewise even when $\dot{\sigma}(z)$ has zeros.

So far we have restricted our analysis to single-scalar models, and at first sight it might seem unlikely that the main result could generalize to models with an arbitrary number of scalars and an arbitrary potential for them. However, a simple argument shows that it *does* generalize, at least locally. The key observation [5] is that for any domain-wall solution, the functions $\Phi(z)$ define a curve in the scalar field target space, and this curve may be chosen as one of the "axes" of a new set of curvilinear coordinates on the target space, in which case the equations defining the curve state that all scalar fields but one, call it $\sigma$, are constant. On this curve the potential is a function only of $\sigma$ and the problem is thus reduced to the one already solved, except of course that the change of target space coordinates needed to achieve this may not be valid globally. However, our result, that all flat or adS-sliced domain-wall solutions are supersymmetric, remains true locally.

Although our Killing spinor results were derived assuming real triplet superpotential, inspired by $D = 5$ supergravity, they are valid for any $D$. We could have obtained these results by considering a simpler Killing spinor equation with a complex superpotential, such as one would find in $D = 4$ by dimensional reduction of the $D = 5$ case (although the discussion to follow on cosmology would then be more involved). No new possibilities can arise from considering more general superpotentials (as confirmed by the results of Ref. [10] for a 5-vector superpotential) because novelty for our purposes would require $k = 1$ and there is no physically acceptable supersymmetric extension of the dS isometry algebra. It is therefore satisfying that, with the exception of the dS foliations of Minkowski or adS (for which the isometry algebra is enlarged), we have not found any supersymmetric $k = 1$ domain walls (although this does not preclude the possibility of first-order equations [11]).

We conclude with some comments on the cosmology case. Recall that any domain-wall solution has an associated cosmological solution with flipped signs of $V$ and $k$. At first it appears that such solutions cannot have Killing spinors because $\Gamma_{\underline{z}}$ now squares to minus the identity, so (23) has no nonzero solutions. However, we must also take $\mathbf{W} \rightarrow i\mathbf{W}$ in order to flip the signs of $V$ and $k$, as is clear from (18) and (21). We now have what appears to be a Killing spinor for any $k \geq 0$ cosmological solution, but the $\mathbf{W} \rightarrow i\mathbf{W}$ step replaces the initial Hermitian mass matrix $\mathbf{W} \cdot \boldsymbol{\tau}$ in the gamma-traced Killing spinor equation by an anti-Hermitian one. As explained earlier, this means that we no longer have a *bona fide* Killing spinor, although we do have what might be called a pseudo Killing spinor. It is unclear what the implications of the existence of pseudo Killing spinors are, but their existence nevertheless explains why $k \geq 0$ FLRW cosmologies are also driven by first-order equations. The implications of this fact remain to be explored.

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