Integrable Nonlinear Evolution Partial Differential Equations in 4 + 2 and 3 + 1 Dimensions

A. S. Fokas*

Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Cambridge, CB3 OWA, United Kingdom

(Received 23 January 2006; published 19 May 2006)

The derivation and solution of integrable nonlinear evolution partial differential equations in three spatial dimensions has been the holy grail in the field of integrability since the late 1970s. The celebrated Korteweg–de Vries and nonlinear Schrödinger equations, as well as the Kadomtsev-Petviashvili (KP) and Davey-Stewartson (DS) equations, are prototypical examples of integrable evolution equations in one and two spatial dimensions, respectively. Do there exist integrable analogs of these equations in three spatial dimensions? In what follows, I present a positive answer to this question. In particular, I first present integrable generalizations of the KP and DS equations, which are formulated in four spatial dimensions and which have the novelty that they involve complex time. I then impose the requirement of real time, which implies a reduction to three spatial dimensions. I also present a method of solution.

DOI: 10.1103/PhysRevLett.96.190201 PACS numbers: 02.30.Ik

Integrable systems have had a significant impact on both theory and phenomenology. Important physical applications range from fluid mechanics and nonlinear optics to quantum gravity and field theories. The modern history of integrable systems begins with the solution of the initial-value problem of the Korteweg–de Vries (KdV) equation using the inverse scattering method [1] and with the formulation of integrable equations as the compatibility condition of two linear eigenvalue equations called a Lax pair [2]. Milestones in the history of integrable *evolution* partial differential equations (PDEs) were the solution of the nonlinear Schrödinger (NLS) equation using the Riemann-Hilbert formalism [3] and the extension of the inverse scattering from one to two spatial dimensions using the nonlocal Riemann-Hilbert [4] and the d-bar formalisms [5–7]. Integrable nonlinear evolution PDEs possess particular localized solutions which dominate the behavior of generic solutions after a long time [8]. Important examples of such solutions are *solitons* for equations in one spatial dimensions, as well as *lumps* [4] and *dromions* [9] for equations in two spatial dimensions.

Gel'fand and the author [10] have emphasized that the inverse scattering method, as well as its extension to two spatial dimensions, can be thought of as nonlinear Fourier transform methods, where the relevant transform pairs are constructed by analyzing the t-independent part of the Lax pair. In what follows, I first present a general approach for constructing nonlinear Fourier transform pairs in *four* dimensions. An example of such a pair can be constructed by analyzing the following eigenvalue equation:

$$\mu_{\bar{x}} + \sigma_3\mu_{\bar{y}} - k[\sigma_3, \mu] + Q\mu = 0, \qquad x = \frac{(\xi + \eta)}{2}, \qquad y = \frac{(\xi - \eta)}{2}, \tag{1}$$

where the complex variables k, ξ, η are given by

$$k = k_R + ik_I, \qquad \xi = \xi_1 + i\xi_2, \qquad \eta = \eta_1 + i\eta_2,$$

with $k_R, k_I, \xi_1, \xi_2, \eta_1, \eta_2$ real variables; the 2×2 matrices σ_3, Q are defined by

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad Q = \begin{pmatrix} 0 & q_1(\xi_1, \xi_2, \eta_1, \eta_2) \\ q_2(\xi_1, \xi_2, \eta_1, \eta_2) & 0 \end{pmatrix},$$

the bar denotes complex conjugation, [,] denotes the usual matrix commutator, and μ is a 2×2 matrix-valued function which depends on the six real variables $\{\xi_1, \xi_2, \eta_1, \eta_2, k_R, k_I\}$.

Indeed, let $\{q_1, q_2\}$ be complex-valued functions with sufficient smoothness which decay for large values of the spatial variables. A nonlinear Fourier transform of $\{q_1, q_2\}$ denoted by $\{f_1, f_2\}$, which depends on the four real variables $\{k_R, k_I, \lambda_R, \lambda_I\}$, is defined by

$$\begin{aligned} f_1 &= \left(\frac{2}{\pi}\right)^3 \int_{\mathbb{R}^4} E_1 q_1 \mu_{22} dV, \\ f_2 &= \left(\frac{2}{\pi}\right)^3 \int_{\mathbb{R}^4} E_2 q_2 \mu_{11} dV, \\ dV &= d\xi_1 d\xi_2 d\eta_1 d\eta_2, \\ E_1 &= \exp[-4i(k_I\xi_1 - k_R\xi_2 + \lambda_I\eta_1 - \lambda_R\eta_2)], \\ E_2 &= \exp[-4i(-\lambda_I\xi_1 + \lambda_R\xi_2 - k_I\eta_1 + k_R\eta_2)], \end{aligned} \tag{2}$$

μ_{ij} denote the components of the matrix μ, and μ is defined in terms of $\{q_1, q_2\}$ through the solution of Eq. (1) supplemented with the boundary condition that μ tends to the identity matrix for large values of the spatial variables. The inverse nonlinear Fourier transform associated with Eqs. (2), i.e., the solution of Eqs. (2) for $\{q_1, q_2\}$ in terms of $\{f_1, f_2\}$, is given by

$$q_1 = \int_{\mathbb{R}^4} E_1^{-1} f_1 \mu_{11} dv, \qquad q_2 = -\int_{\mathbb{R}^4} E_2^{-1} f_2 \mu_{22} dv, \tag{3}$$

where $dv = dk_R dk_I d\lambda_R d\lambda_I$ and μ is defined through

$\{f_1, f_2\}$ via the following nonlocal d-bar problem:

$$\frac{\partial\mu(k_R, k_I)}{\partial\bar{k}} = \int_{\mathbb{R}^2} \mu(\lambda_R, \lambda_I)F(\lambda_R, \lambda_I, k_R, k_I)d\lambda_R d\lambda_I,$$

$$\mu = \mathrm{diag}(1,1) + O\left(\frac{1}{k}\right), \qquad k \to \infty,$$

where F is an off diagonal 2×2 matrix with "12" and "21" entries equal to f_1E_1 and f_2E_2, and for convenience of notation we have suppressed the dependence of μ and F on the space variables $\xi_1, \xi_2, \eta_1, \eta_2$.

If $\{q_1, q_2\}$ are sufficiently small, then μ_{11} and μ_{22} are approximated by 1, and hence $\{f_1, f_2\}$ reduce to the usual Fourier transforms of $\{q_1, q_2\}$ in the four independent variables $\{\xi_1, \xi_2, \eta_1, \eta_2\}$ with associated Fourier exponents $\{k_I, -k_R, \lambda_I, -\lambda_R\}$ and $\{-\lambda_I, \lambda_R, -k_I, k_R\}$, respectively. Similarly, Eqs. (3) reduce to the usual inverse Fourier transform in four dimensions.

The derivations of the nonlinear Fourier transform pair (2) and (3) involves the *spectral analysis* of Eq. (1). Namely, we first construct an appropriate solution μ of (1) in terms of $\{q_1, q_2\}$. Using the fact that this μ is bounded for all complex values of k, we then derive an alternative representation for μ by computing $\partial\mu/\partial\bar{k}$ and employing Pompieu's formula [11]. This alternative representation is expressed in terms of the functions $\{f_1, f_2\}$ defined by Eqs. (2), and, hence, equating these two representations of μ, we find the inverse formulas (3).

It turns out that nonlinear Fourier transform pairs in four dimensions, such as the pair (2) and (3), can be used for the solution of the initial-value problem of integrable nonlinear evolution PDEs in four spatial dimensions, provided that we allow time to be complex. For example, using (2) and (3) we can solve the following system of integrable nonlinear evolution PDEs for the two independent variables $\{q_1, q_2\}$, in four real spatial variables $\{\xi_1, \xi_2, \eta_1, \eta_2\}$, and in two real time variables $\{t_1, t_2\}$,

$$(-1)^j\partial_{\bar{t}}q_j + \tfrac{1}{4}(\partial_{\bar{\xi}}^2 + \partial_{\bar{\eta}}^2)q_j - q_j\partial_{\bar{\xi}}^{-1}(q_1q_2)_{\bar{\eta}} = 0, \qquad \partial_{\bar{\xi}}^{-1}f = -\frac{1}{\pi}\int_{R^2}\frac{f(\xi_1', \xi_2')}{\xi - \xi'}d\xi_1 d\xi_2, \tag{4}$$

where, in the first of Eqs. (4), j is 1 or 2 and $t = t_1 + it_2$. Indeed, let $\{q_1^{(0)}, q_2^{(0)}\}$ denote the initial conditions $\{q_j^{(0)}\}_1^2 = \{q_j(\xi_1, \xi_2, \eta_1, \eta_2, 0, 0)\}_1^2$. Let $\{f_1^{(0)}, f_2^{(0)}\}$ be the functions defined by Eq. (2) with $\{q_1, q_2\}$ replaced by $\{q_1^{(0)}, q_2^{(0)}\}$. Let $\{q_j(\xi_1, \xi_2, \eta_1, \eta_2, t_1, t_2)\}_1^2$ be defined by Eqs. (3), with f_1 and f_2 replaced by $f_1^{(0)}E$ and $f_2^{(0)}E^{-1}$, where

$$E = \exp[4i(\lambda_R\lambda_I + k_Rk_I)t_1 - 2i(\lambda_R^2 - \lambda_I^2 + k_R^2 - k_I^2)t_2].$$

Then $\{q_1, q_2\}$ solve Eqs. (4) with the initial conditions $\{q_1^{(0)}, q_2^{(0)}\}$.

If $\{q_1^{(0)}, q_2^{(0)}\}$ are sufficiently small, then μ_{11} and μ_{22} can be approximated by 1 and$\{q_1, q_2\}$ solve Eqs. (4) without the nonlinear term.

After understanding the crucial importance of the complexification of time, it is possible to employ the so-called *dressing method* introduced by Zakharov and his coworkers (see, for example, [12]), which is technically much easier than the nonlinear Fourier transform method. This method has the disadvantage that it *cannot* be used for the solution of the initial-value problem, but it has the advantage that it constructs integrable nonlinear PDEs, as well as large classes of their solutions, in a straightforward, essentially algebraic manner. For example, using the dressing method, we can derive the following result. Let $\mu(k_R, k_I)$ be the solution of the nonlocal d-bar problem

$$\frac{\partial\mu(k_R, k_I)}{\partial\bar{k}} = \int_{\mathbb{R}^2} \mu(\lambda_R, \lambda_I)F(k_R, k_I, \lambda_R, \lambda_I)d\lambda_R d\lambda_I,$$

$$\mu(k_R, k_I) = 1 + \frac{\nu}{k} + O\left(\frac{1}{k^2}\right), \qquad k \to \infty, \tag{5}$$

where

$$\begin{aligned} F = f(k_R, k_I, \lambda_R, \lambda_I)\exp\{4i[(\lambda_I - k_I)x_1 + (k_R - \lambda_R)x_2 \\ + 2(\lambda_R\lambda_I - k_Rk_I)y_1 + (\lambda_I^2 - \lambda_R^2 + k_R^2 - k_I^2)y_2 \\ + (k_I^3 - \lambda_I^3 + 3\lambda_I\lambda_R^2 - 3k_R^2k_I)t_1 + (k_R^3 - \lambda_R^3 \\ + 3\lambda_R\lambda_I^2 - 3k_Rk_I^2)t_2]\}, \end{aligned} \tag{6}$$

and for convenience of notation we have suppressed in Eqs. (5) the dependence of μ, F, and ν on the six real variables $\{x_1, x_2, y_1, y_2, t_1, t_2\}$. Assume that the above μ is unique and that q is defined in terms of μ by $q = 2\nu_{\bar{x}}$. Then q solves the generalized Kadomtsev-Petviashvili (KP) equation

$$q_{\bar{t}} = \tfrac{1}{4}q_{\bar{x}\bar{x}\bar{x}} - \tfrac{3}{2}qq_{\bar{x}} + \tfrac{3}{4}\partial_{\bar{x}}^{-1}q_{\bar{y}\bar{y}}, \tag{7}$$

where $x = x_1 + ix_2, \quad y = y_1 + iy_2, \quad t = t_1 + it_2$. Furthermore, this equation possesses the Lax pair

$$\begin{aligned} (D_y - D_x^2 + q)\mu = 0, \\ [D_t - D_x^3 + \tfrac{3}{2}qD_x + \tfrac{3}{4}(q_{\bar{x}} + \partial_{\bar{x}}^{-1}q_{\bar{y}})]\mu = 0, \end{aligned} \tag{8}$$

where D_x, D_y, D_t are defined by

$$D_x = \partial_{\bar{x}} + k, \qquad D_y = \partial_{\bar{y}} + k^2, \qquad D_t = \partial_{\bar{t}} + k^3.$$

Indeed, if D denotes any of these operators, then $D\mu$ satisfies the first of Eqs. (5) provided that F satisfies

$$F_{\bar{x}} = (\lambda - k)F, \qquad F_{\bar{y}} = (\lambda^2 - k^2)F,$$
$$F_{\bar{t}} = (\lambda^3 - k^3)F.$$

The solution of these equations and the requirement of boundness yield the expression for F defined by (6). Let $L\mu$ denote the left-hand side of the first of Eqs. (8). Then $L\mu$ satisfies the first of Eqs. (5), and, furthermore, because of the definition of q, it satisfies $L\mu = O(1/k)$ as $k \to \infty$; thus, $L\mu = 0$. Similarly, if $M\mu$ denote the left-hand side of the second of Eqs. (8), then $M\mu = 0$. The compatibility condition of Eqs. (8) yields Eq. (7).

The dressing method provides a straightforward tool for constructing large classes of explicit solutions. For example, the particular choice $f = \sum_1^n c_j \delta(k - k_j)\delta(\lambda - \lambda_j)$, where δ denotes the complex Dirac function, yields an explicit soliton-type solution of Eq. (7). In a similar manner, the following explicit solution of Eqs. (4) can be constructed:

$$q_1 = \frac{-2(k-\lambda)c_1 E_1 E}{1 - c_1 c_2 E_1 E_2}, \qquad q_2 = \frac{2(k-\lambda)c_2 E_2 E^{-1}}{1 - c_1 c_2 E_1 E_2}, \tag{9}$$

where E_1, E_2, E have been defined earlier and c_1, c_2 are arbitrary complex constants. For appropriate values of these constants, these solutions are bounded and they represent soliton-type solutions of the system of nonlinear PDEs defined by Eqs. (4).

One of the most important features of integrable equations is that they possess the so-called Painlevé property [13]. The investigation of this property for integrable evolution PDEs involves an expansion in the complex-t plane, which is usually carried out formally by simply treating t as a complex variable. However, the results presented here suggest a systematic procedure for constructing the proper complexification of integrable PDEs. For example, such complexifications for the KdV and KP equations are Eq. (7) with $x = y$ and Eq. (7), respectively. Similarly, proper complexifications of the NLS and Davey-Stewartson (DS) equations are Eqs. (4) with $\xi = \eta$ and Eqs. (4), respectively.

Equation (7) reduces to the KP equation provided that q is independent of t_2, x_2, and y_2. The exponential appearing in (6) implies that this is the case provided that $\lambda_R = k_R$ and $\lambda_I = -k_I$, in which case the nonlocal d-bar problem (5) becomes the local d-bar problem $\partial\mu/\partial\bar{k} = \mu(k_R, -k_I)\tilde{F}(k_R, k_I)$. Similarly, it is straightforward to reduce Eqs. (4) to the DS equation.

After the pioneering work of Ref. [12], nonlocal d-bar problems have been considered by several authors. However, until now they have been used for the solution of evolution PDEs in two spatial variables; thus, they were effectively reduced to local d-bar problems.

We next discuss the reduction to 3 + 1. For the generalized KP equation, this reduction is explicit, yielding a single evolution PDE in three spatial dimensions, whereas for the generalized DS it is implicit, yielding an evolution PDE in four spatial dimensions constrained by an additional equation.

The solution of the generalized KP equation is independent of t_2 provided that the coefficient of t_2 in the exponential (6) vanishes. This implies a constraint between the four spectral variables $\{k_R, k_I, \lambda_R, \lambda_I\}$, which in turn implies a constraint between the four spatial variables. Hence, by eliminating the t_2 dependence, Eq. (7) becomes an evolution equation in only three independent spatial variables.

Assuming q is real, suppressing the t_2 dependence, and renaming t_1 as t, Eq. (7) yields

$$\begin{aligned} q_t &= \tfrac{1}{16}(\partial_{x_1}^3 - 3\partial_{x_1}\partial_{x_2}^2)q - \tfrac{3}{2}qq_{x_1} + \tfrac{3}{8}(\partial_{y_1}^2 - \partial_{y_2}^2)\,\mathrm{Re}\{\partial_{\bar{x}}^{-1}q\} \\ &\quad - \tfrac{3}{4}\partial_{y_1}\partial_{y_2}\,\mathrm{Im}\{\partial_{\bar{x}}^{-1}q\}, \\ 0 &= \tfrac{1}{16}(-\partial_{x_2}^3 + 3\partial_{x_2}\partial_{x_1}^2)q - \tfrac{3}{2}qq_{x_2} \\ &\quad + \tfrac{3}{8}(\partial_{y_1}^2 - \partial_{y_2}^2)\,\mathrm{Im}\{\partial_{\bar{x}}^{-1}q\} + \tfrac{3}{4}\partial_{y_1}\partial_{y_2}\,\mathrm{Re}\{\partial_{\bar{x}}^{-1}q\}. \end{aligned} \tag{10}$$

Letting $\partial_{\bar{x}}^{-1}q = R + iI$, where R and I are real, and applying the Laplacian operator $\Delta = \partial_x\partial_{\bar{x}}/4$ to this equation, we find $\Delta R = 2q_{x_1}$ and $\Delta I = -2q_{x_2}$. Replacing in Eq. (10) R and I by $2\Delta^{-1}q_{x_1}$ and $-2\Delta^{-1}q_{x_2}$, respectively, differentiating the first of Eqs. (10) by ∂_{x_2}, the second by ∂_{x_1}, and adding the resulting equations, we find the first of Eqs. (11) below; similarly, differentiating the first of Eqs. (10) by ∂_{x_1}, the second of Eqs. (10) by ∂_{x_2}, and subtracting the resulting equations, we find the second of Eqs. (11) below:

$$\begin{aligned} \frac{\partial^2 q}{\partial t \partial x_1} &= \frac{1}{4}(\partial_{x_1}^3\partial_{x_2} - \partial_{x_2}^3\partial_{x_1})q - \frac{3}{2}\partial_{x_1}\partial_{x_2}(q^2) + \frac{3}{2}\partial_{y_1}\partial_{y_2}q, \\ \frac{\partial^2 q}{\partial t \partial x_2} &= \frac{1}{16}(\partial_{x_1}^4 + \partial_{x_2}^4 - 6\partial_{x_1}^2\partial_{x_2}^3)q + \frac{3}{4}(\partial_{x_2}^2 - \partial_{x_1}^2)(q^2) \\ &\quad + \frac{3}{2}(\partial_{y_1}^2 - \partial_{y_2}^2)q. \end{aligned} \tag{11}$$

Integrating the first of Eqs. (11) with respect to y_1, we find an explicit expression for q_{y_2} in terms of derivatives of q with respect to t, x_1, x_2, y_1, as well as integrals with respect to y_1. Using this expression to compute $q_{y_2y_2}$, and then substituting the resulting formula in the second of Eqs. (11), we find a *single* evolution equation for q involving only the variables t, x_1, x_2, y_1.

Finally, we discuss the reduction of Eqs. (4). The solution of these equations is independent of t_1 provided that $\lambda_I\lambda_R + k_I k_R = 0$, which in turn implies a constraint between the four spatial variables. For sufficiently small initial conditions, this constraint is $\partial_{\xi_1}\partial_{\xi_2} + \partial_{\eta_1}\partial_{\eta_2} = 0$, but, in general, it takes a nonlinear form: Letting $q_2 = c\bar{q}_1$, where c is a complex constant, suppressing the t_1 dependence, and renaming q_1 and t_2 as q and t, Eqs. (4) become

$$\begin{aligned} iq_t - \tfrac{1}{2}(\partial_{\xi_1}^2 - \partial_{\xi_2}^2 + \partial_{\eta_1}^2 - \partial_{\eta_2}^2)q + 2q\,\mathrm{Re}\{c\partial_{\bar{\xi}}^{-1}|q|_{\bar{\eta}}^2\} &= 0, \\ (\partial_{\xi_1}\partial_{\xi_2} + \partial_{\eta_1}\partial_{\eta_2})q - 2q\,\mathrm{Im}\{c\partial_{\bar{\xi}}^{-1}|q|_{\bar{\eta}}^2\} &= 0. \end{aligned} \tag{12}$$

In contrast to the analogous case of Eqs. (10), we have *not* been able to solve the second of Eqs. (12) explicitly.

It can be verified that Eqs. (12) possess the explicit soliton-type localized solution given by the first of

Eqs. (9), with the restrictions

$$\lambda_I \lambda_R + k_I k_R = 0 \quad \text{and} \quad |c_1|^2 c = \frac{k - \lambda}{\bar{k} - \bar{\lambda}}.$$

The so-called Lowner system [14] is the t-independent part of the Lax pair of several integrable equations in 2 + 1, and, therefore, its proper complexification should be investigated. Similar considerations are valid for the PDEs considered in Refs. [15–18].

The existence of integrable nonlinear equations in four spatial dimensions involving complex time suggests that perhaps the idea of complexifying time should be investigated in the context of modern field theories.

*Electronic address: t.fokas@damtp.cam.ac.uk

[1] C. S. Gardner, J. M. Greene, M. D. Kruskal, and R. M. Miura, Phys. Rev. Lett. **19**, 1095 (1967).
[2] P. D. Lax, Comm. Pure Appl. Math. **21**, 467 (1968).
[3] V. E. Zakharov and A. B. Shabat, Sov. Phys. JETP **34**, 62 (1972).
[4] A. S. Fokas and M. J. Ablowitz, Stud. Appl. Math. **69**, 211 (1983).
[5] M. J. Ablowitz, D. BarYaacov, and A. S. Fokas, Stud. Appl. Math. **69**, 135 (1983).
[6] A. S. Fokas and L. Y. Sung, Inverse Probl. **8**, 673 (1992).
[7] R. Beals and R. Coifman, Inverse Probl. **5**, 87 (1989).
[8] P. Deift, A. Its, and X. Zhou, in *Important Developments in Soliton Theory*, edited by A. S. Fokas and V. E. Zakharov (Springer, Berlin, 1993).
[9] M. Boiti, J. Leon, L. Martina, and F. Pempinelli, Phys. Lett. A **132**, 432 (1988); A. S. Fokas and P. Santini, Phys. Rev. Lett. **63**, 1329 (1989).
[10] A. S. Fokas and I. M. Gel'fand, Lett. Math. Phys. **32**, 189 (1994).
[11] M. J. Ablowitz and A. S. Fokas, *Introduction and Applications of Complex Variables* (Cambridge University Press, Cambridge, England, 2003), 2nd ed.
[12] V. E. Zakharov and S. V. Manakov, Funct. Anal. Appl. **19**, 89 (1985).
[13] B. Grammaticos, B. Dorizzi, and A. Ramani, Phys. Rev. Lett. **53**, 1 (1984); A. Ramani, B. Grammaticos, and T. Bountis, Phys. Rep. **180**, 159 (1989).
[14] B. G. Konopelchenko and C. Rogers, J. Math. Phys. (N.Y.) **34**, 214 (1993).
[15] R. Radha and M. Lakshmanan, J. Phys. A **29**, 1551 (1996).
[16] P. G. Estevez and J. Prada, J. Nonlinear Math. Phys. **11**, 164 (2004).
[17] A. N. Leznov and M. V. Saveliev, Commun. Math. Phys. **74**, 111 (1980).
[18] K. Tenenblat and G. L. Terng, Ann. Math. **111**, 477 (1980).