Classification Scheme for Phenomenological Universalities in Growth Problems in Physics and Other Sciences

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A classification in universality classes of broad categories of phenomenologies, belonging to physics and other disciplines, may be very useful for a cross fertilization among them and for the purpose of pattern recognition and interpretation of experimental data. We present here a simple scheme for the classification of nonlinear growth problems. The success of the scheme in predicting and characterizing the well known Gompertz, West, and logistic models, suggests to us the study of a hitherto unexplored class of nonlinear growth problems.

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Present efforts towards the understanding of complex systems in physics, biology, economics, and social science require complementary microscopic and macroscopic descriptions. Microscopic models depend on such a large number of parameters that they often lose almost any predictive power, even when the calculations do not become forbiddingly difficult or time consuming. On the other hand, macroscopic descriptions are often inadequate and do not take advantage of the enormous progress that has been achieved at the microscopic level in recent years. An intermediate (mesoscopic) approach [1–3] may be very fruitful, but a bridging among the various levels [4] is not always easy to accomplish.

A different approach has consequently emerged for the treatment of problems , which do not directly require a detailed description of the system to be investigated. The idea is to exploit the spectacular advancement of interdisciplinary research, which has taken place in the last two decades or so, involving, e.g., the relevance of scale laws, complexity, and nonlinearity in virtually all disciplines.

In this context many patterns have been discovered, which are remarkably similar, although they concern completely different phenomenologies. This is hardly surprising, since often the ‘‘background’’ mathematics is the same. We shall call them ‘‘phenomenological universalities’’ [5], in the sense that they refer to a ‘‘transversal’’ generality (not to a uniformly general behavior within a given class of phenomena).

As examples of universality we can quote the ‘‘life's universal scaling laws’’ [6], which will be discussed later, and the ‘‘universality of nonclassical nonlinearity’’ [7]. The latter suggests that unexpected effects, such as those recently discovered by P. Johnson and collaborators [8] and called by them ‘‘fast dynamics,’’ may be found as well, although possibly with quite different manifestations, in other fields of research.

A reliable macroscopic analysis of a complex system requires two fundamental ingredients: nonlinearity and stochasticity. Nonlinearity is more fundamental because the stochastic behavior requires a nonlinear dynamics. Therefore nonlinearity must be considered as the fundamental feature of these systems and in this Letter we consider general growth problems based on this crucial aspect. We shall show that different ‘‘degrees of nonlinearity’’ (as specified below) correspond to various growth patterns, which can be systematically classified.

For this purpose, let us consider the very broad class of growth phenomena, which may be described by the simple law

$$\frac{dY(t)}{dt} = \alpha(t)Y(t), \tag{1}$$

where $\alpha(t)$ represents the specific growth rate, which may vary with time, of a given variable $Y(t)$. By introducing the nondimensional variables $\tau = \alpha(0)t$, $y(t) = Y(t)/Y(0)$, and $a(\tau) = \alpha(t)/\alpha(0)$, Eq. (1) becomes:

$$\frac{dy(\tau)}{d\tau} = a(\tau)y(\tau), \tag{2}$$

with $y(0) = a(0) = 1$. By defining the time variation of $a(\tau)$ through a function $\Phi(a)$

$$\Phi(a) = -\frac{da(\tau)}{d\tau} \tag{3}$$

we obtain a system of two differential equations, which may generate a variety of growth patterns, according to the explicit form of $\Phi(a)$ (see also Ref. [9]), and is usually analyzed by the standard fixed points and characteristic curves methods [10,11].

In this contribution we are not directly interested in this aspect, but we wish to show, instead, how the nonlinear terms in $\Phi(a)$ affect the growth dynamics process.

We assume that $|a(\tau)| < 1$, for $\tau > 0$, and consider a polynomial expansion for $\Phi(a)$

$$\Phi(a) = \Sigma_{n=0}^{\infty} b_n a^n \tag{4}$$

in which we retain only a limited number of terms. Borrowing from the language of phase transitions [12], we define as belonging to the phenomenological universality class of order N (which we shall call UN, $N = 1, 2, \ldots$), the ensemble of all the phenomenology, which may be suitably described by truncating the series at the power $n = N$. In the following we shall analyze in detail the classes $U1$, $U2$, and $U3$ and provide a description of their nonlinear properties.

The "linear" behavior of the system corresponds to a constant specific growth rate, i.e., $\Phi(a) = 0$ (or $b_n = 0$ for any n). Then $y(\tau)$ follows a purely exponential law. Also the case $b_0 \neq 0$ with all $b_n = 0$ for $n \geq 1$, can be easily shown to lead to an exponential growth. Since we are interested only in the nonlinear effects, we shall assume $b_0 = 0$. This does not cause any loss of generality, since one can always expand Φ in the variable $\beta = a - c$, where c is a solution of $\Sigma_{n=0}^{\infty} b_n c^n = 0$. In the β expansion the coefficient of β^0 vanishes. Likewise, again without any loss of generality, we can set $b_1 = 1$, as one would have from an expansion in the variable $\gamma = a/b_1$.

In order to study the various classes of universality and obtain the corresponding differential equations and solutions, we write from Eqs. (2) and (3)

$$-\Phi(a)\frac{dy}{da} = ay, \tag{5}$$

from which it follows

$$\ln y = -\int \frac{a da}{\Phi(a)} + \text{const.} \tag{6}$$

By solving the previous equation with respect to the variable $a(\tau)$ and then substituting into Eq. (2), one obtains the differential equation characterizing the class. The integration constant can be easily obtained from the initial conditions.

Let us then start by considering the class $U1$, i.e., with $N = 1$. From Eq. (6) and $\Phi(a) = a$, it immediately follows

$$\frac{dy}{d\tau} = y - y\ln y, \tag{7}$$

with the solution

$$y = \exp[1 - \exp(-\tau)]. \tag{8}$$

Equation (7) represents the "canonical" form of $U1$ differential equations and corresponds to the Gompertz law, originally introduced [13] in actuarial mathematics to evaluate the mortality tables and, nowadays, largely applied to describe economical and biological growth phenomena. For example, the Gompertz law gives a very good phenomenological description of tumor growth [14,15] and can be related to the energetic cellular balance [16]. It is remarkable that it does not contain any free parameter (except for the scale and linear parameters which have not been included, as discussed before), i.e., all Gompertz curves are (under the mentioned proviso) identical.

Let us now turn our attention to the class $U2$. From Eqs. (6) and (3) and $\Phi(a) = a + ba^2$, where $b = b_2$, it follows

$$\frac{dy}{d\tau} = \alpha_2 y^p - \beta_2 y, \tag{9}$$

where $\alpha_2 = (1 + b)/b$, $p = 1 - b$, and $\beta_2 = 1/b$ with the solution

$$y = [1 + b - b\exp(-\tau)]^{1/p}. \tag{10}$$

By identifying y with the mass of a biological system, $y = m$ (with $m_0 = y_0 = 1$), and defining the asymptotic mass

$$M = \lim_{\tau \to \infty} m(\tau) = (1 + b)^{1/p} \tag{11}$$

it is easy to show that Eqs. (9) and (10) correspond to the well-known allometric West equation for the case $p = 3/4$ [17]. In their ontogenetic growth model, m represents the mass of any living organism, from protozoa to mammalians (including plants as well). By redefining their mass and time variables $z = 1 - (y/m)^b$ and $\theta = -\tau + \ln b - b\ln M$ they obtain the very elegant parameterless universal law

$$z = \exp(-\theta), \tag{12}$$

which fits well the data for a variety of different species, ranging from shrimps to hens to cows. It is interesting to note that, in a subsequent work [18], West and collaborators give an interpretation of θ as the "biological clock," based on the organism's internal temperature.

An extension of West's law to neoplastic growths has been recently suggested by Guiot, Delsanto, Deisboeck, and collaborators [19,20]. Although an unambigous fitting of experimental data is much harder for the tumors (except for cultures of multicellular tumor spheroids), the extension seems to work well. Of course, other mechanisms must be taken into account, such as the pressure from the surrounding tissue [21]. Another important issue is the actual value of the exponent p, which has been the object of a strong debate [22]. Recently Guiot *et al.* [23] proposed that p may vary dynamically with the fractal nature of the input channels (e.g., at the onset of angiogenesis).

Although it is not obvious from a comparison between Eqs. (7) and (9), $U1$ represents a special case ($b = 0$) of $U2$, as it obviously follows from the power expansion of Φ (which has $b = 0$ in $U1$). This can be verified directly by carefully performing the limit for $b \to 0$ in Eq. (10). In fact, it is interesting to plot y vs τ in a sort of phase diagram (see Fig. 1) in which the Gompertz curve (solid line) separates the two $U2$ regions, "West-like" and "logistic-like" ($b > 0$ and $b < 0$, respectively).

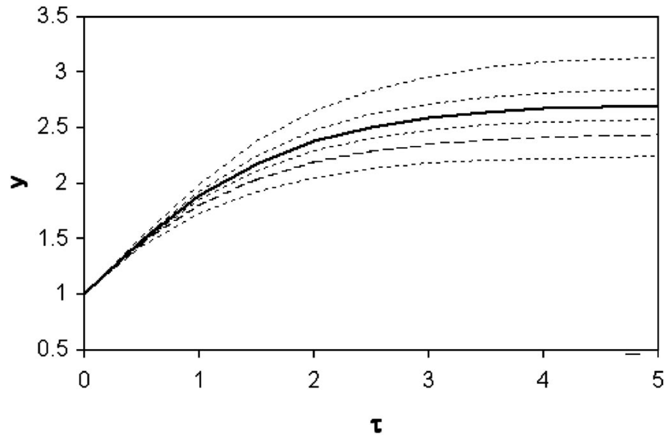

FIG. 1. Growth curves belonging to the class $U2$. From the top to the bottom the values of the parameter b are $-0.25, -0.1, 0.1, 0.25, 0.5$, respectively: The solid curve ($b = 0$, $p = 1$) corresponds to the Gompertzian ($U1$), while the dashed one refers to the value proposed in [6] $p = 3/4$ ($b = 1/4$).

This leads to a very suggestive interpretation of Eq. (9). Having added a term to the $\Phi(a)$ expansion, we gain, in $U2$, the possibility of adding a ''new'' ingredient, which turns out to be a different dimensionality of the ''energy flux'', i.e., input, output, and consumption (metabolism). Thus the first term on the right-hand side of Eq. (9) may be related [24] to the premise that the tendency of natural selection to optimize energy transport has led to the evolution of fractal-like distribution networks with an exponent p for their terminal units vs an exponent 1 for flux mechanisms related to the total number of cells. When $b = 0$, $p = 1$, we lose the new ingredient and fall back into $U1$.

This is confirmed also by considering the logistic equations, corresponding to Eq. (9) with negative b. The usual logistic equation is obtained for $p = 2$. As well known in population dynamics [25], here the new ingredient is the competition for resources.

Finally, we consider the class $U3$. Writing

$$\Phi(a) = a(1 + ba + ca^2) \tag{13}$$

from Eq. (6) it follows

$$\int \frac{da}{1 + ba + ca^2} = K - \ln y. \tag{14}$$

In this case there are three subclasses, $U31$, $U32$, and $U33$, corresponding to $\Delta = 4c - b^2 \geq <0$. For brevity we limit ourselves to report here the canonical equation for $U31$:

$$\frac{dy}{d\tau} = \alpha_3 y - \beta_3 y^p + \gamma_3 \frac{dy^p}{d\tau}, \tag{15}$$

where $d = \sqrt{-\Delta}$, $p = 1 - d$, $K = (d - b - 2c)/(d + b + 2c)$, $\alpha_3 = (d - c)/2c$, $\beta_3 = K(d + c)/2c$, and $\gamma_3 = -K/(1 - d)$. It is interesting to observe that, in the same way that $U2$ adds (with respect to $U1$) a term with a different dimensionality to the energy flux contribution, $U3$ adds such a term [the last one in Eq. (15)] to the growth part.

To conclude, we have developed a simple scheme that allows the classification in nonlinear phenomenological universality classes of all the growth problems, which can be described by Eqs. (2) and (3). We have found that the first class $U1$ corresponds to the Gompertz curve, which has no free parameters (apart from scale and linear ones). The second class $U2$ includes all the West-like and logistic curves and has a free parameter b: when $b = 0$ we fall back into $U1$ (Gompertz).

The success of the scheme in obtaining the classes $U1$ and $U2$, when one or two terms are retained in the expansion of $\Phi(a)$, has suggested to us to investigate the class $U3$, which is generated by simply adding one more term [see Eq. (13)]. To our knowledge, this class has never been investigated before. A remarkable result is that each new class adds a new ''ingredient'' (or growth mechanism). E.g., $U2$ allows for the possible presence of two dimensionalities in the energy flux. $U3$ extends such a possibility to the growth term (the time derivative).

In addition to its intrinsic elegance [26], the concept of universality classes may be useful for several reasons of applicative relevance. In fact it greatly facilitates the cross fertilization among different fields of research. Also, if an unexpected effect is found experimentally in a field, similar effects ''*mutatis mutandis*'' should also be sought in similar, although unrelated, experiments in other fields. Finally, if a detailed study is performed to recognize the patterns that are characteristics of the most relevant classes (and subclasses), this could greatly help in classifying and fitting new sets of experimental data independently of the field of application. The proposed formalism is presently being applied to growth problems of current interest in physics. Exploitations of the cross fertilization ''strategy'' are extremely important, in particular, for the export of models and methods which have been developed in physics to other disciplines (and vice versa).

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