# Holographic Derivation of Entanglement Entropy from the anti–de Sitter Space/Conformal Field Theory Correspondence

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A holographic derivation of the entanglement entropy in quantum (conformal) field theories is proposed from anti–de Sitter/conformal field theory (AdS/CFT) correspondence. We argue that the entanglement entropy in $d+1$ dimensional conformal field theories can be obtained from the area of $d$ dimensional minimal surfaces in $AdS_{d+2}$, analogous to the Bekenstein-Hawking formula for black hole entropy. We show that our proposal agrees perfectly with the entanglement entropy in 2D CFT when applied to $AdS_3$. We also compare the entropy computed in $AdS_5 \times S^5$ with that of the free $\mathcal{N}=4$ super Yang-Mills theory.

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One of the most remarkable successes in gravitational aspects of string theory is the microscopic derivation of the Bekenstein-Hawking entropy $S_{BH}$,

$$S_{BH} = \frac{\text{area of horizon}}{4G_N}, \tag{1}$$

for Bogomolnyi-Prasad-Sommerfield black holes [1]. This idea relates the gravitational entropy with the degeneracy of quantum field theory as its microscopic description. Taking the horizon limit, we can regard this as a special example of anti–de Sitter/conformal field theory (AdS/CFT) correspondence [2–4]. It claims that the $d+1$ dimensional conformal field theories ($CFT_{d+1}$) are equivalent to the (super)gravity on $d+2$ dimensional anti–de Sitter space $AdS_{d+2}$. We expect that each CFT is sitting at the boundary of AdS space.

On the other hand, there is a different kind of entropy called entanglement entropy (von Neumann entropy) in quantum mechanical systems. The entanglement entropy, $S_A = -\mathrm{tr}_A \rho_A \log \rho_A$, $\rho_A = \mathrm{tr}_B |\Psi\rangle\langle\Psi|$, provides us with a convenient way to measure how closely entangled (or how "quantum") a given wave function $|\Psi\rangle$ is. Here, the total system is divided into two subsystems $A$ and $B$, and $\rho_A$ is the reduced density matrix for the subsystem $A$ obtained by taking a partial trace over the subsystem $B$ of the total density matrix $\rho = |\Psi\rangle\langle\Psi|$. Intuitively, we can think of $S_A$ as the entropy for an observer who is accessible only to the subsystem $A$ and cannot receive any signals from $B$. In this sense, the subsystem $B$ is analogous to the inside of a black hole horizon for an observer sitting in $A$, i.e., outside of the horizon. Indeed, an original motivation of entanglement entropy was its similarity to the Bekenstein-Hawking entropy [5,6].

The entanglement entropy is of growing importance in many fields of physics in our exploration for a better understanding of quantum systems. For example, in a modern trend of condensed matter physics it has been becoming clear that quantum phases of matter need to be characterized by their pattern of entanglement encoded in many-body wave functions of ground states, rather than conventional order parameters [7–9]. Recently, the entanglement entropy has been extensively studied in low-dimensional quantum many-body systems as a new tool to investigate the nature of quantum criticality (refer to [10] and references therein, for example).

For 1D quantum many-body systems at criticality (i.e., 2D CFT), it is known that the entanglement entropy is given by [10,11]

$$S_A = \frac{c}{3} \log\left[\frac{L}{\pi a} \sin\left(\frac{\pi l}{L}\right)\right], \tag{2}$$

where $l$ and $L$ are the length of subsystem $A$ and total system $A \cup B$ (both ends of $A \cup B$ are periodically identified), respectively; $a$ is a ultraviolet (UV) cutoff (lattice spacing); $c$ is the central charge of the CFT. When we are away from criticality, Eq. (2) is replaced by [7,10]

$$S_A = \frac{c}{6} \mathcal{A} \log \frac{\xi}{a}, \tag{3}$$

where $\xi$ is the correlation length and $\mathcal{A}$ is the number of boundary points of $A$ [e.g., $\mathcal{A} = 2$ in the setup of (2)].

In spite of these recent developments, and its similarity to the black hole entropy, a comprehensive gravitational interpretation of the entanglement entropy has been lacking so far. Here, we present a simple proposal on this issue in the light of AdS/CFT duality. Earlier discussions from different viewpoints can be found in, e.g., papers [12,13]. Define the entanglement entropy $S_A$ in a CFT on $\mathbb{R}^{1,d}$ (or $\mathbb{R} \times S^d$) for a subsystem $A$ that has an arbitrary $d-1$ dimensional boundary $\partial A \in \mathbb{R}^d$ (or $S^d$). In this setup we propose the following "area law":

$$S_A = \frac{\text{area of } \gamma_A}{4G_N^{(d+2)}}, \tag{4}$$

where $\gamma_A$ is the $d$ dimensional static minimal surface in $AdS_{d+2}$ whose boundary is given by $\partial A$, and $G_N^{(d+2)}$ is the $d+2$ dimensional Newton constant. Intuitively, this suggests that the minimal surface $\gamma_A$ plays the role of a holographic screen for an observer who is accessible only to the subsystem $A$. We show explicitly the relation (4) in the lowest dimensional case $d = 1$, where $\gamma_A$ is given by a geodesic line in $AdS_3$. We also compute $S_A$ from the gravity side for general $d$ and compare it with field theory results, which is successful at least qualitatively. From (4), the basic properties of the entanglement entropy, (i) $S_A = S_B$ ($B$ is the complement of $A$) and (ii) $S_{A_1} + S_{A_2} \geq S_{A_1 \cup A_2}$ (subadditivity), are obvious.

We can also define the entanglement entropy at finite temperature $T = \beta^{-1}$. For example, in a 2D CFT on a infinitely long line, it is given by replacing $L$ in Eq. (2) with $i\beta$ [10]. We argue that Eq. (4) still holds in $T > 0$ cases. Note that $S_A = S_B$ is no longer true if $T > 0$ since $\rho$ is in a mixed state generically. At high temperature, we will see that this occurs due to the presence of black hole horizon in the dual gravity description.

Let us start with the entanglement entropy in 2D CFTs. According to AdS/CFT correspondence, gravitational theories on $AdS_3$ space of radius $R$ are dual to $(1 + 1)$D CFTs with the central charge [14]

$$c = \frac{3R}{2G_N^{(3)}}. \tag{5}$$

The metric of $AdS_3$ in the global coordinate $(t, \rho, \theta)$ is

$$ds^2 = R^2(-\cosh\rho^2 dt^2 + d\rho^2 + \sinh\rho^2 d\theta^2). \tag{6}$$

At the boundary $\rho = \infty$ of $AdS_3$ the metric is divergent. To regulate physical quantities we put a cutoff $\rho_0$ and restrict the space to the bounded region $\rho \leq \rho_0$. This procedure corresponds to the UV cutoff in the dual CFTs [15]. If $L$ is the total length of the system with both ends identified, and $a$ is the lattice spacing (or UV cutoff) in the CFTs, we have the relation (up to a numerical factor)

$$e^{\rho_0} \sim L/a. \tag{7}$$

The $(1 + 1)$D spacetime for the $CFT_2$ is identified with the cylinder $(t, \theta)$ at the boundary $\rho = \rho_0$. The subsystem $A$ is the region $0 \leq \theta \leq 2\pi l/L$. Then $\gamma_A$ in Eq. (4) is identified with the static geodesic that connects the boundary points $\theta = 0$ and $2\pi l/L$ with $t$ fixed, traveling inside $AdS_3$ [Fig. 1(a)]. With the cutoff $\rho_0$ introduced above, the geodesic distance $L_{\gamma_A}$ is given by $\cosh(\frac{L_{\gamma_A}}{R}) = 1 + 2\sinh^2\rho_0 \sin^2\frac{\pi l}{L}$.

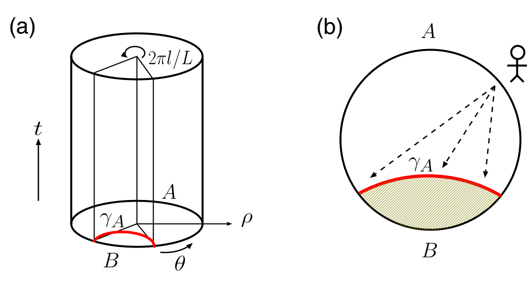


FIG. 1 (color online). (a) $AdS_3$ space and $CFT_2$ living on its boundary, and (b) a geodesics $\gamma_A$ as a holographic screen.

Assuming the large UV cutoff $e^{\rho_0} \gg 1$, the entropy (4) is expressed as follows, using Eq. (5):

$$S_A \simeq \frac{R}{4G_N^{(3)}} \log\left(e^{2\rho_0}\sin^2\frac{\pi l}{L}\right) = \frac{c}{3}\log\left(e^{\rho_0}\sin\frac{\pi l}{L}\right). \tag{8}$$

This entropy precisely coincides with the known CFT result (2) after we remember the relation Eq. (7).

This proposed relation (4) suggests that the geodesic (or minimal surface in the higher dimensional case) $\gamma_A$ is analogous to an event horizon as if $B$ were a black hole, though the division into $A$ and $B$ is actually artificial. In other words, the observer, who is not accessible to $B$, will probe $\gamma_A$ as a holographic screen [16], instead of $B$ itself [Fig. 1(b)]. The minimal surface provides the severest entropy bound when we fix its boundary condition. In our case it saturates the bound.

More generally, we can consider a subsystem $A$ which consists of multiple disjoint intervals as follows:

$$A = \{x | x \in [r_1, s_1] \cup [r_2, s_2] \cup \cdots \cup [r_N, s_N]\}, \tag{9}$$

where $0 \leq r_1 < s_1 < r_2 < s_2 < \cdots < r_N < s_N \leq L$. In the dual $AdS_3$ description, the region (9) corresponds to $\theta \in \cup_{i=1}^N [\frac{2\pi r_i}{L}, \frac{2\pi s_i}{L}]$ at the boundary. In this case it is not straightforward to determine minimal (geodesic) lines responsible for the entropy. However, we can find the answer from the entanglement entropy computed in the CFT side. The general prescription of calculating the entropy for such systems is given in [10] using conformal mapping. For our system (9), we find, when rewritten in the $AdS_3$ language, the following expression of $S_A$:

$$S_A = \frac{\sum_{i,j} L_{r_j,s_i} - \sum_{i<j} L_{r_j,r_i} - \sum_{i<j} L_{s_j,s_i}}{4G_N^{(3)}}, \tag{10}$$

where $L_{a,b}$ is the geodesic distance between two boundary points $a$ and $b$. We can think that the correct definition of minimal surface is given by the numerator in Eq. (10).

Next we turn to the entanglement entropy at finite temperature. We assume the spacial length of the total system $L$ is infinitely long such that $\beta/L \ll 1$. At high temperature, the gravity dual of the CFT is the Euclidean Banados-Teitelboim-Zanelli (BTZ) black hole [17] with the metric given by

$$ds^2 = (r^2 - r_+^2)d\tau^2 + \frac{R^2}{r^2 - r_+^2}dr^2 + r^2 d\varphi^2. \tag{11}$$

The Euclidean time is compactified as $\tau \sim \tau + \frac{2\pi R}{r_+}$ to obtain a smooth geometry in addition to the periodicity $\varphi \sim \varphi + 2\pi$. Looking at its boundary, we find the relation $\frac{\beta}{L} = \frac{R}{r_+} \ll 1$ between the CFT and the BTZ black hole.

The subsystem $A$ is defined by $0 \leq \varphi \leq 2\pi l/L$ at the boundary. Then we expect that the entropy can be computed from the geodesic distance between the boundary points $\varphi = 0, 2\pi l/L$ at a fixed time. To find the geodesic line, it is useful to remember the familiar fact that the Euclidean BTZ black hole at temperature $T_{BTZ}$ is equivalent to the thermal $\mathrm{AdS}_3$ at temperature $1/T_{\mathrm{BTZ}}$. This equivalence can be interpreted as a modular transformation in the CFT side [18]. If we define the new coordinates $r = r_+ \cosh\rho$, $r_+\tau = R\theta$, and $r_+\varphi = Rt$, then the metric (11) indeed becomes the Euclidean version of $\mathrm{AdS}_3$ metric (6). Thus the geodesic distance can be found in the same way as the zero-temperature case, $\cosh(L_{\gamma_A}/R) = 1 + 2\cosh^2\rho_0 \sinh^2(\frac{\pi l}{\beta})$, where the UV cutoff is interpreted as $e^{\rho_0} \sim \beta/a$. Then the area law (4) again reproduces the known CFT result [10] $S_A(\beta) = \frac{c}{3}\log[\frac{\beta}{\pi a}\sinh(\pi l/\beta)]$. In the multi-interval cases we find the same formula (10).

It is instructive to repeat the same analysis in the Poincaré metric $ds^2 = R^2 z^{-2}(dz^2 - dx_0^2 + dr^2)$. We define the subsystem $A$ by the region $-l/2 \le r \le l/2$ at the boundary $z = 0$. The geodesic line $\gamma_A$ is given by

$$(r, z) = \frac{l}{2}(\cos s, \sin s), \qquad (\epsilon \le s \le \pi - \epsilon), \tag{12}$$

where the infinitesimal $\epsilon$ is the UV cutoff $\epsilon \sim 2a/l$ (or equally $z_{\mathrm{UV}} \sim a$). We obtain the entropy $S_A$ as follows: $S_A = \frac{L_{\gamma_A}}{4G_N^{(3)}} = \frac{R}{2G_N^{(3)}}\int_\epsilon^{\pi/2}\frac{ds}{\sin s} = \frac{c}{3}\log\frac{l}{a}$. This reproduces the small $l$ limit of Eq. (2) [11].

When we perturb a CFT by a relevant perturbation, the renormalization group flow generically drives the theory to a trivial IR fixed point. We denote the correlation length $\xi$ in the latter theory. In the AdS dual, this massive deformation corresponds to capping off the IR region, restricting the allowed values of $z$ to $z \le \xi$. In the large $l$ limit, we find $S_A = \frac{1}{4G_N^{(3)}}\int_\epsilon^{2\xi/l}\frac{ds}{\sin s} = \frac{c}{6}\log\frac{\xi}{a}$. This agrees with the CFT result with $\mathcal{A} = 1$ (3) [7,10].

Motivated by the success in our gravitational derivation of the entanglement entropy for $d = 1$, it is interesting to extend the idea to higher dimensional cases ($d \ge 2$). A natural thing to do is to replace geodesic lines with minimal surfaces. The computations are analogous to the evaluation of Wilson loops [19], though the dimension of relevant minimal surfaces is different.

We will work in the Poincaré metric for $\mathrm{AdS}_{d+2}$,

$$ds^2 = R^2 z^{-2}\left(dz^2 - dx_0^2 + \sum_{i=1}^{d} dx_i^2\right). \tag{13}$$

We consider two examples for the shape of $A$. The first one is a straight belt $A_S = \{x_i | x_1 \in [-l/2, l/2], x_{2,3,\ldots,d} \in [-\infty, \infty]\}$ at the boundary $z = 0$ [Fig. 2(a)]. In this case

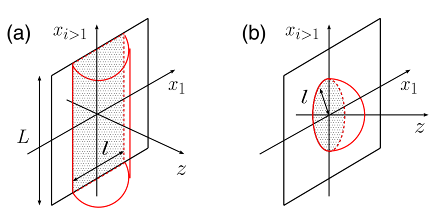


FIG. 2 (color online). Minimal surfaces in $\mathrm{AdS}_{d+2}$: (a) $A_S$ and (b) $A_D$.

the minimal surface is defined by $dz/dx_1 = \sqrt{z_*^{2d} - z^{2d}}/z^d$, where $z_*$ is determined by $l/2 = \int_0^{z_*} dz z^d (z_*^{2d} - z^{2d})^{-1/2} = z_*\sqrt{\pi}\Gamma(\frac{d+1}{2d})/\Gamma(\frac{1}{2d})$. The area of this minimal surface is

$$\mathrm{area}_{A_S} = \frac{2R^d}{d-1}\left(\frac{L}{a}\right)^{d-1} - \frac{2^d \pi^{d/2} R^d}{d-1}\left(\frac{\Gamma(\frac{d+1}{2d})}{\Gamma(\frac{1}{2d})}\right)^d \left(\frac{L}{l}\right)^{d-1}, \tag{14}$$

where $L$ is the length of $A_S$ in the $x_{2,3,\ldots,d}$ direction.

The second example is the disk $A_D$ defined by $A_D = \{x_i | r \le l\}$ [Fig. 2(b)] in the polar coordinate $\sum_i dx_i^2 = dr^2 + r^2 d\Omega_{d-1}^2$. The minimal surface is a $d$ dimensional ball $B^d$ defined by (12). Its area is

$$\begin{aligned}\mathrm{area}_{A_D} &= C\int_{a/l}^{1} dy \frac{(1-y^2)^{(d-2)/2}}{y^d} \\ &= p_1 (l/a)^{d-1} + p_3 (l/a)^{d-3} \\ &\quad + \begin{cases} p_{d-1}(l/a) + p_d + O(a/l), & d\text{: even}, \\ p_{d-2}(l/a)^2 + q\log(l/a) + O(1), & d\text{: odd}, \end{cases}\end{aligned} \tag{15}$$

where $C = 2\pi^{d/2}R^d/\Gamma(d/2)$ and $p_1/C = (d-1)^{-1}$, etc. For $d$ even, $p_d/C = (2\sqrt{\pi})^{-1}\Gamma(\frac{d}{2})\Gamma(\frac{1-d}{2})$ and for $d$ odd, $q/C = (-)^{(d-1)/2}(d-2)!!/(d-1)!!$.

From these results, the entanglement entropy can be calculated by Eq. (4). Each of (14) and (15) has a UV divergent term $\sim a^{-d+1}$ that is proportional to the area of the boundary $\partial A$. This agrees with the known area law of the entanglement entropy in quantum field theories [5,6]. Note that this area law is related to our Eq. (4) via the basic property of holography.

We may prefer a physical quantity that is independent of the cutoff (i.e., universal). The second term in Eq. (14) has this property. In general, when $A$ is a finite size, there is a universal and conformal invariant constant contribution to $S_A$ if $d$ is even (see [20] for properties of minimal surfaces in AdS). In $(2+1)$D topological field theories the constant contribution to $S_A$ encodes the quantum dimension and is called the topological entanglement entropy [8,9]. If $d$ is odd, the coefficient of the logarithmic term $\sim\log(l/a)$ is universal as in Eq. (2).

Let us apply the previous results to a specific string theory setup. Type IIB string on $\mathrm{AdS}_5 \times S^5$ is dual to 4D $\mathcal{N} = 4$ $SU(N)$ super Yang-Mills theory [2]. The radii of $\mathrm{AdS}_5$ and $S^5$ are given by the same value $R = (4\pi g_s \alpha'^2 N)^{1/4}$. The 5D Newton constant is related to the 10D one via $G_N^{(10)} = \pi^3 R^5 G_N^{(5)}$. Then we obtain from Eqs. (14) and (15)

$$S_{A_S} = \frac{N^2 L^2}{2\pi a^2} - 2\sqrt{\pi}\left(\frac{\Gamma(\frac{2}{3})}{\Gamma(\frac{1}{6})}\right)^3 \frac{N^2 L^2}{l^2}, \tag{16}$$

$$S_{A_D} = N^2\left[\frac{l^2}{a^2} - \log\left(\frac{l}{a}\right) + O(1)\right]. \tag{17}$$

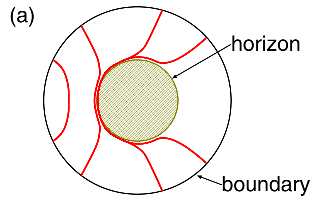


FIG. 3 (color online). (a) Minimal surfaces $\gamma_A$ for various sizes of $A$. (b) $\gamma_A$ and $\gamma_B$ wrap the different parts of the horizon.

It is interesting to compare the finite universal term in Eq. (16) with the field theory one. For free real scalars and fermions in general dimensions, one way to compute $S_{A_S}$ is presented in [21] (see also [22]). Indeed, this leads to the same behavior in $a$ and $l$ as in Eq. (16). Following this approach, we can estimate finite contributions from 6 scalars and 4 Majorana fermions in the $\mathcal{N} = 4$ Yang-Mills multiplet. In the end, we obtain $S_{\text{finite}}^{\text{freeCFT}} \sim -(0.068 + g)N^2L^2/l^2$, where $g$ is the contribution from the gauge field ($g = 0.010$ if we treat the gauge field as 2 scalars). On the other hand, our AdS$_5$ result (16) leads to $S_{\text{finite}}^{\text{AdS}} \sim -0.051N^2L^2/l^2$. We may think this is a good agreement if we remember that the gravity description corresponds to the strongly coupled gauge theory instead of the free theory as in [23].

As the final example, we discuss the $\mathcal{N} = 4$ super Yang-Mills theory on $\mathbb{R}^3$ at a finite temperature $T$, which is dual to the AdS black hole defined by the metric [24] $ds^2 = R^2[\frac{du^2}{hu^2} + u^2(-hdt^2 + dx_1^2 + dx_2^2 + dx_3^2) + d\Omega_5^2]$, where $h = 1 - u_0^4/u^4$, $u_0 = \pi T$. For the straight belt $A_S$, the area is given by (putting the cut off $u \sim z^{-1} \sim a$) $\text{area}_{A_S} = 2R^3L^2 \int_{u_*}^{a^{-1}} \frac{du u^6}{\sqrt{(u^4 - u_0^4)(u^6 - u_*^6)}}$, where $u_*$ satisfies $l/2 = \int_{u_*}^{\infty} du[(u^4 - u_0^4)(u^6/u_*^6 - 1)]^{-1/2}$. This contains the UV divergence $\sim a^{-2}$ as before. As in the analogous computation of Wilson loops [25], we also expect a term which is proportional to the area of $A$. Indeed, when $l$ is large ($u_* \sim u_0$) we find the constant term $\sim \pi^3 R^3 L^2 l T^3$. This leads to the finite part of $S_A$

$$S_{\text{finite}} \simeq \frac{\pi^2}{2} N^2 T^3 L^2 l = \frac{\pi^2 N^2 T^3}{2} \times (\text{area of } A_S). \quad (18)$$

We can regard this entropy as a part of the Bekenstein-Hawking entropy of black 3-branes [23], which is proportional to the area of horizon situated at $u = u_0$. Thus we can interpret the part (18) as a thermal entropy contribution to the total entanglement entropy at finite temperature. In our gravitational description, this part arises because the minimal surface wraps a part of the black hole horizon [Fig. 3(a)]. If we expand the size of $A$ until it coincides with the total system (in the global coordinate), $\gamma_A$ wraps the horizon completely and $S_A$ becomes equal to the Bekenstein-Hawking entropy as expected. In a sense, the overall normalization in Eq. (4) is fixed from Eq. (1) once we consider the entanglement entropy at finite temperature. Note that at finite temperature $S_A = S_B$ does not hold generically, as is clear from Fig. 3(b).

As argued in [13,26], the AdS black hole can be dual to an entanglement of two different CFTs at the two boundaries. As a specific limit, we may think the black hole entropy is the same as the entanglement entropy of the CFTs as the minimal surface wrap the horizon.

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