

Rayleigh-Plateau and Gregory-Laflamme Instabilities of Black Strings

Vitor Cardoso*

Department of Physics and Astronomy, The University of Mississippi, University, Mississippi 38677-1848, USA

Óscar J. C. Dias†

Perimeter Institute for Theoretical Physics, Waterloo, Ontario N2L 2Y5, Canada and Department of Physics, University of Waterloo, Waterloo, Ontario N2L 3G1, Canada

(Received 4 February 2006; published 8 May 2006)

Many and very general arguments indicate that the event horizon behaves as a stretched membrane. We explore this analogy by associating the Gregory-Laflamme instability of black strings with a classical membrane instability known as the Rayleigh-Plateau instability. We show that the key features of the black string instability can be reproduced using this viewpoint. In particular, we get good agreement for the threshold mode in all dimensions and exact agreement for large spacetime dimensionality. The instability time scale is also well described within this model, as well as the dimensionality dependence. It also predicts that general nonaxisymmetric perturbations are stable. We further argue that the instability of ultraspinning black holes follows from this model.

DOI: [10.1103/PhysRevLett.96.181601](https://doi.org/10.1103/PhysRevLett.96.181601)

PACS numbers: 11.25.Wx, 04.70.Dy, 47.10.A–

The existence of black holes is perhaps the most dramatic prediction of Einstein's theory, the very concept of which makes full use of the nonlinearity of the equations and also of our notion of space and time. Despite being apparently very complex objects, black holes can be associated with many of the familiar quantities of everyday physics. The first major breakthrough in this direction was hinted at by Bekenstein [1], who conjectured that black holes are endowed with thermodynamic properties, namely, with an entropy proportional to its area. That black holes are thermodynamics entities was established once and for all by Hawking [2], who verified explicitly that black holes radiate, and therefore have an associated temperature. This analogy carries over to higher-dimensional scenarios, which seem to be a prerequisite for consistency in many modern theories. In the general higher-dimensional gravity theories, there are other objects called black branes [3], with event horizons. These are basically extended black holes: the horizon, instead of having the topology of a sphere, can have, for instance, the topology of sphere times a line—a cylinder. The analogy with thermodynamics can still be formulated, and in fact one can even go further and formulate an analogy with hydrodynamics [4]. The thermodynamic description, both for black holes and black branes, is based on the four laws of black hole dynamics formulated by Bardeen, Carter, and Hawking [5]. The first law (we will take for simplicity uncharged, static objects) describes how a black hole, characterized by its mass M , horizon area A , and $T = 1/(32\pi M)$, evolves when we throw an infinitesimal amount of matter into it:

$$dM = TdA. \quad (1)$$

The second law states that in any classical process the horizon area must increase, $dA \geq 0$. It is very tempting

to associate these two laws with the first and second laws of thermodynamics, respectively, in which case T would be proportional to a temperature and A to an entropy (the other two laws of black hole mechanics also have a correspondence with the zero and fourth thermodynamic laws). The final ingredient to proceed consistently with this association was given by Hawking [2], who realized that black holes are indeed radiating objects and that one can indeed associate them with a temperature $T_H = 4T$.

We can also argue that Eq. (1) can be looked at as a law for fluids, with T being an effective surface tension [6]. Regarding the event horizon as a kind of fluid membrane is a position adopted in the past [7]. The first works in black hole mechanics actually considered T as a surface tension (see the work by Smarr [8] and references therein), which is rather intuitive: in fluids the potential energy, associated with the storage of energy at the surface, is indeed proportional to the area. Later, Thorne and co-workers [7] developed the “membrane paradigm” for black holes, envisioning the event horizon as a kind of membrane with well-defined mechanical, electrical, and magnetic properties. Not only is this a simple picture of a black hole, it is also useful for calculations and understanding what black holes are really like. There are other instances where a membrane behavior seems evident: Eardley and Giddings [9], studying high-energy black hole collisions, found a soap-bubble-like law for the process, while many modern interpretations for black hole entropy and gravity “freeze” the degrees of freedom in a lower dimensional space, in what is known as holography [10]. In Ref. [4] it was shown that a membrane approach works surprisingly well, yielding precisely the same results as the anti-de Sitter/conformal field theory correspondence. There have also been attempts to work the other way round: computing liquid surface tension from the (analog) black hole entropy [11].

We consider here an interesting class of black objects without spherical topology, the black branes, and treat them as fluids with a surface tension subjected to (1) and without gravity. A broad class of these objects are unstable against gravity, in a mechanism known as Gregory-Laflamme instability [12]. This is a gravitational instability along the “extended” dimension, pictured in Fig. 1. We will show that we can mimic most aspects of this instability using the fluid analogy.

Take a black string with radius R_0 , and extended along a direction z . Then the Gregory-Laflamme mechanism makes any small perturbation with wavelength λ of the order of, or larger than, the radius of the cylinder R_0 grow exponentially with time. This is a very robust long-wavelength instability.

The details for the instability can be found in the original papers [12]. For wavelengths larger than a threshold λ_c , the instability appears. In the general setup of black strings with $D - 1$ spatial directions with radius R_0 and a transverse direction z , one finds [12] that the threshold wave number increases with dimension number D and so does the maximum growth time scale. For very large number of dimensions, the threshold mode behaves as [13]

$$k_c \equiv 2\pi/\lambda_c \sim \sqrt{D}/R_0, \quad (2)$$

Quite remarkably, we can find a similar instability for fluids with surface tension, as shown in Plateau’s celebrated study [14] on the stability of bodies under the influence of surface tension. He established a fundamental result of classical continuum mechanics: a cylinder longer than its circumference is energetically unstable to breakup. This result was put on a firmer basis by Rayleigh [15], who computed the exact instability time scale for the problem. The reasoning for the appearance of an instability is the following: consider a small disturbance of a long cylinder (we take its axis as the z axis) of fluid with radius R_0 and height z . Considering a small axisymmetric perturbation along the surface of the cylinder, we write for the disturbed cylinder

$$r(z) = R_0 + \epsilon R_1 \cos(kz) + \epsilon^2 R_2, \quad (3)$$

where ϵ measures the perturbed quantities (R_2 is a second order quantity, and its usefulness will be understood shortly). The volume of this cylinder can be easily com-

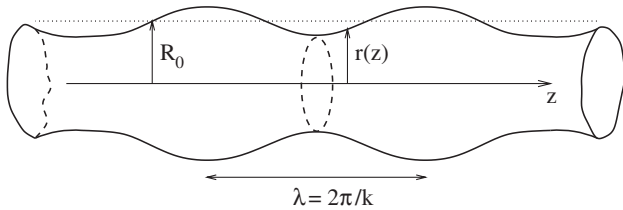


FIG. 1. Black strings and fluid cylinders are unstable to perturbations on the extended dimension, i.e., along the axis of the cylinder. Ripples propagating along this axis grow exponentially with time for wavelengths of order of the radius of the cylinder.

puted to be

$$V = z\pi \left[R_0^2 + \epsilon^2 \left(2R_0 R_2 + \frac{R_1^2}{2} \right) \right] + \mathcal{O}(\epsilon^3). \quad (4)$$

If we impose constant density one must have $R_2 = -\frac{R_1^2}{4R_0}$. With this condition, the surface area of the disturbed cylinder is

$$A = z\pi \left[2R_0 + \frac{\epsilon^2 R_1^2}{2R_0} (k^2 R_0^2 - 1) \right]. \quad (5)$$

The potential energy per unit length is therefore

$$P = \frac{\pi \epsilon^2 R_1^2}{2R_0} (k^2 R_0^2 - 1) T. \quad (6)$$

We conclude that the system is unstable for $k < 1/R_0$, since in this case a perturbation of the form (3) decreases the potential energy.

One can further show that nonaxisymmetric perturbations, with profile $r(z, \phi) = R_0 + \epsilon R_1 \cos(kz) \cos(m\phi) + \epsilon^2 R_2$ (where m is an integer that identifies the angular mode) are stable for any $m \neq 0$ [15]. The reason is that the potential energy for these nonaxisymmetric modes is given by $P = \frac{\pi \epsilon^2 R_1^2}{2R_0} (k^2 R_0^2 - 1 + m^2) T$, which never decreases for $m \neq 0$.

We can generalize the Rayleigh-Plateau construction to a general number of dimensions. Take a hyper-cylinder with $D - 1$ spatial directions with radius R_0 and a transverse direction z (for the previous example $D = 3$). The axisymmetric threshold wave number is

$$R_0 k_c = \sqrt{D - 2} \quad \text{with} \quad k_c \equiv 2\pi/\lambda_c. \quad (7)$$

For wave numbers smaller (larger wavelengths) than this critical value the cylinder is unstable. Moreover, as in the original Rayleigh-Plateau situation, only symmetric modes seem to be unstable; nonaxisymmetric modes are in general stable. Therefore, the Rayleigh-Plateau instability, like the Gregory-Laflamme instability, should disappear for modes other than the s modes. This was recently conjectured by Hovdebo and Myers [16] using an argument based on the relation between the thermodynamic and the Gregory-Laflamme instabilities [17]. Kudoh [18] has explicitly verified that the Gregory-Laflamme instability only affects s modes.

To motivate quantitatively the suggested association between the Rayleigh-Plateau and the Gregory-Laflamme instabilities, it is important to compare the dependence of the threshold wave number kR_0 on dimension D , for both instabilities. This is done by comparing (2) with (7). In Table I, we list the value of the Rayleigh-Plateau threshold mode $R_0 k_c$ for several dimensions D . We also list the threshold wave number for the Gregory-Laflamme instability, with values taken from [13]. There is good agreement between them. In the large D limit there is exact agreement: both the Gregory-Laflamme and the Rayleigh-Plateau critical wave number behave as $k_c R_0 \sim \sqrt{D}$. This

TABLE I. Dimensionless threshold wave number kR_0 for the Rayleigh-Plateau instability of a higher-dimensional fluid cylinder, and the corresponding threshold wave number for the Gregory-Laflamme instability (data taken from Ref. [13]).

D spatial dimensions	4	5	6	7	8	9	49	99
Rayleigh-Plateau	1.41	1.73	2.00	2.24	2.45	2.66	6.78	9.80
Gregory-Laflamme	0.876	1.27	1.58	1.85	2.09	2.30	6.72	9.75

is, we think, a nontrivial check on the conjecture that black branes behave as fluid membranes with surface tension. We can further compare the evolution of the instability time scale with its wavelength, and study the dependence of the instability time scale on the spacetime dimension. We compute the Rayleigh-Plateau instability time scale using fluid dynamics, following Rayleigh [15]. For a 3-dimensional cylinder, and assuming the perturbation goes as $R_1 \sim e^{\Omega t}$, Rayleigh gets the following expression for Ω :

$$\Omega^2 = \frac{T}{\rho R_0^3} \frac{ikR_0 J'_0(ikR_0)}{J_0(ikR_0)} (1 - k^2 R_0^2), \quad (8)$$

with J being a Bessel function. We have generalized Rayleigh's procedure for higher dimensions, and the results are shown in Fig. 2. We assumed an effective surface tension and density associated to the temperature and energy density of a D -dimensional Schwarzschild black hole. The qualitative behavior of the instability time scale matches surprisingly well that of the Gregory-Laflamme [12] one. In particular, note that (i) the maximum growth rate grows with the number of spacetime dimensions, and (ii) the corresponding wave number also grows with dimension number D . Moreover, the location of the threshold wave number is exactly as predicted with the energy argument [see Eq. (6)]. There is one discrepancy only: the maximum instability for the Rayleigh-Plateau case, i.e., the maximum Ω , is approximately 1 order of magnitude larger than the maximum of the Gregory-Laflamme. This could be due to the complete neglect of gravity in the outside of the cylinder (in the Rayleigh-Plateau analogy). Indeed if one included gravity effects, a redshift was bound to occur, thereby lowering Ω [19].

It is interesting to ask what is the endpoint of the Rayleigh-Plateau instability. In four dimensions, the Rayleigh-Plateau instability makes a cylinder pinch off, giving rise to drop formation (see for instance the original works [14,15] or the review work by Eggers [20]). To see that this is expected, consider a $D = 3$ cylinder breaking up at a Rayleigh-Plateau length, i.e., a cylinder of height $z = \lambda_c = 2\pi R_0$. Can this give rise to a drop of the same volume? Same volume implies that the radius of the final spherical drop would be $1.6765R_0$. This makes the surface area of the drop smaller than the corresponding for the cylinder, $S_{\text{drop}} = 0.8947S_{\text{cyl}}$, and thus it makes sense that a drop is the final state. Now, if we play this geometrical game in higher dimensions, we conclude that there is critical dimension between $D = 10$ and $D = 11$. For $D \geq$

11 a sphere seems not to be the favored endpoint, since it no longer has less surface area. Once again, there is a very similar phenomena in the gravity case (see, e.g., the discussion in Refs. [16,21,22]). Take a black string breaking up at a Gregory-Laflamme length. Sorkin found a critical spacetime dimension ($d = D + 1$) between $d = 13$ and $d = 14$, above which the black string is no longer entropically unstable against the formation of a spherical black hole [21]. It would be interesting to further study this issue. In particular, it is important to understand what is the endpoint of the Rayleigh-Plateau instability for $D \geq 11$, and to find if in the fluid model there is a new branch of solutions that would be the analogue of the nonuniform black string solutions of Ref. [23], and if so to study their stability.

It has been shown by Horowitz and Maeda [24] that pinch-off, if it occurs at all in the black string case, must do so in infinite affine time. This immediately suggests that an attractive endpoint could be nonuniform black strings [23,24]. Now, the breakup of liquid jets with surface tension in the absence of viscosity is known to happen in finite time, but the inclusion of viscosity (which seems a necessary ingredient to model realistic black objects [25]) may change this [20], so even this unexpected feature might be discussed within this analogue model. It is quite amusing that some of the dynamical features of the instability of black strings have already been observed in liquids: the final state of some liquid bridges (finite-size liquid cylinders), unstable under the Rayleigh-Plateau instability, is a nonsymmetric state (see Ref. [26] and its references) and

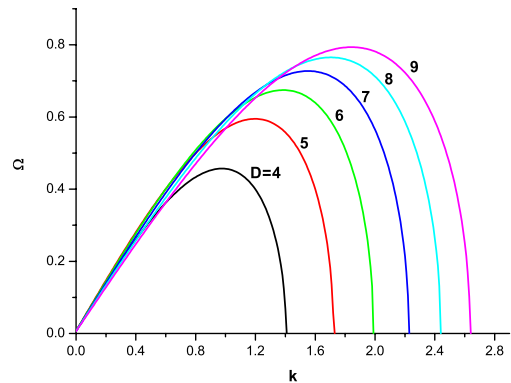


FIG. 2 (color online). Rayleigh-Plateau instability of a hypercylinder in several dimensions. Here the effective surface tension and density were chosen to match those of a higher-dimensional nonrotating black hole.

the breakup of liquid jets is quite generally a self-similar phenomena [20].

The interpretation of T as surface tension can also improve—and strengthen—our understanding of the instability of ultraspinning black holes (Myers-Perry [27] black holes with high rotation), a conjecture recently made by Emparan and Myers in connection with the Gregory-Laflamme instability [28]. Take a slowly rotating black hole. As the rotation rate increases, the surface becomes flatter at the poles until zero Gaussian curvature occurs. But a liquid drop (at least in four dimensions) develops instabilities before zero Gaussian curvature is reached [29]. Assuming a correspondence between rotating liquid drops and rotating black holes, one may well expect ultraspinning black holes to be unstable. We note that this very same reasoning was applied by Smarr [8] many years ago. In four dimensions, it seems that the upper Kerr bound in the angular momentum is small enough to avoid the development of such instabilities. However, in dimensions higher than six, rotating black holes have no Kerr-like bound, and the instability might well set in.

It is possible that an event horizon behaves dynamically much as a fluid interface without gravity, as we have shown. If this is so, the second law of black hole dynamics should be something like a soap-bubble law: sphericity is preferred for it is the minimum energy shape (which would justify why most stable solutions in GR have spherical topology). Finally, there are many ways to extend these results, by including additional effects, like charge or rotation in the problem. One would naively expect either of these to increase the instability, but a more careful study has to be done. Take rotation, for instance. The centrifugal force, scaling with r , contributes with a destabilizing effect, since there is an increased pressure under a crest but a reduced pressure under a trough. So, in principle, the threshold wavelength should be smaller [30]. However, there is no simple reasoning (as far as we know) to tell the effects of rotation on the strength of the perturbation, i.e., on the instability time scale. This is partly because the effective surface tension of black hole itself depends on rotation, decreasing with increasing rotation. This would be an interesting direction for further study.

We would like to thank the participants of the KITP Program: “Scanning New Horizons: GR Beyond 4 Dimensions,” in particular, E. Berti, L. Bombelli, M. Cavaglià, J. Hovdebo, D. Marolf, R. Myers, E. Sorkin, and T. Wiseman for valuable comments and suggestions. V.C. acknowledges financial support from Fundação Calouste Gulbenkian, and O.D. from FCT (grant No. SFRH/BPD/2004), and from a NSERC Discovery Grant (University of Waterloo). Research at Perimeter Institute is supported in part by funds from NSERC of Canada and MEDT of Ontario. This research was supported by the NSF under Grant No. PHY99-07949.

*Electronic address: vcardoso@phy.olemiss.edu

Also at CFC Universidade de Coimbra, Portugal

†Also at KITP UCSB, Santa Barbara, CA 93106, USA.

Electronic address: odias@perimeterinstitute.ca

- [1] J. D. Bekenstein, Phys. Rev. D **7**, 2333 (1973).
- [2] S. W. Hawking, Commun. Math. Phys. **43**, 199 (1975).
- [3] G. T. Horowitz and A. Strominger, Nucl. Phys. **B360**, 197 (1991).
- [4] P. Kovtun, D. T. Son, and A. O. Starinets, Phys. Rev. Lett. **94**, 111601 (2005); J. High Energy Phys. **10** (2003) 064.
- [5] J. M. Bardeen, B. Carter, and S. W. Hawking, Commun. Math. Phys. **31**, 161 (1973).
- [6] R. Defay, I. Prigogine, A. Bellemans, and D. H. Everett, *Surface Tension and Adsorption* (John Wiley and Sons, New York, 1966).
- [7] K. S. Thorne, D. A. MacDonald, and R. H. Price, *Black Holes: The Membrane Paradigm* (Yale University Press, New Haven, CT, 1986).
- [8] L. Smarr, Phys. Rev. Lett. **30**, 71 (1973).
- [9] D. M. Eardley and S. B. Giddings, Phys. Rev. D **66**, 044011 (2002).
- [10] G. 't Hooft, gr-qc/9310026; L. Susskind, J. Math. Phys. (N.Y.) **36**, 6377 (1995); J. M. Maldacena, Adv. Theor. Math. Phys. **2**, 231 (1998); E. Witten, Adv. Theor. Math. Phys. **2**, 253 (1998); R. Bousso, J. High Energy Phys. **07** (1999) 004.
- [11] D. J. E. Callaway, Phys. Rev. E **53**, 3738 (1996).
- [12] R. Gregory and R. Laflamme, Phys. Rev. Lett. **70**, 2837 (1993); Nucl. Phys. **B428**, 399 (1994).
- [13] B. Kol and E. Sorkin, Classical Quantum Gravity **21**, 4793 (2004).
- [14] J. Plateau, *Statique Expérimentale et Théorique des Liquides Soumis aux Seules Forces Moléculaires* (Gauthier-Villars, Paris, 1873).
- [15] L. Rayleigh, Proc. London Math. Soc. **10**, 4 (1878).
- [16] J. L. Hovdebo and R. C. Myers, Phys. Rev. D **73**, 084013 (2006).
- [17] H. S. Reall, Phys. Rev. D **64**, 044005 (2001).
- [18] H. Kudoh, hep-th/0602001.
- [19] R. C. Myers (private communication).
- [20] J. Eggers, Rev. Mod. Phys. **69**, 865 (1997).
- [21] E. Sorkin, Phys. Rev. Lett. **93**, 031601 (2004).
- [22] M.-I. Park, Classical Quantum Gravity **22**, 2607 (2005).
- [23] S. S. Gubser, Classical Quantum Gravity **19**, 4825 (2002); T. Wiseman, Classical Quantum Gravity **20**, 1137 (2003).
- [24] G. T. Horowitz and K. Maeda, Phys. Rev. Lett. **87**, 131301 (2001).
- [25] M. Parikh and F. Wilczek, Phys. Rev. D **58**, 064011 (1998).
- [26] J. M. Montanero, G. Cabezas, and J. Acero, Phys. Fluids **14**, 682 (2002).
- [27] R. C. Myers and M. J. Perry, Ann. Phys. (N.Y.) **172**, 304 (1986).
- [28] R. Emparan and R. C. Myers, J. High Energy Phys. **09** (2003) 025.
- [29] S. Chandrasekhar, Proc. R. Soc. A **286**, 1 (1964).
- [30] L. E. Johns and R. Narayanan, *Interfacial Instability* (Springer-Verlag, New York, 2002).