Nontrapping Arrest of Langmuir Wave Damping near the Threshold Amplitude

A. V. Ivanov and Iver H. Cairns
School of Physics, The University of Sydney, New South Wales 2006, Sydney, Australia
(Received 13 October 2005; published 1 May 2006)

Evolution of a Langmuir wave is studied numerically for finite amplitudes slightly above the threshold which separates damping from nondamping cases. Arrest of linear damping is found to be a second-order effect due to ballistic evolution of perturbations, resonant power transfer between field and particles, and organization of phase space into a positive slope for the average distribution function $f_{\rm av}$ around the resonant wave phase speed v_ϕ. Near the threshold trapping in the wave potential does not arrest damping or saturate the subsequent growth phase.

DOI: 10.1103/PhysRevLett.96.175001 PACS numbers: 52.35.Mw, 52.35.Sb

Plasma theory has usually been pursued independently of the theory of critical phenomena. Recently, however, it has been revealed that evolution of a monochromatic electrostatic Langmuir wave of finite amplitude in a Maxwellian plasma is a threshold phenomenon. Specifically, after a short initial period of approximately linear damping according to Landau's [1] classic theory, a wave with initial amplitude A_0 greater than a threshold $A_0^\star$ stops decreasing and starts to grow approximately exponentially before undergoing irregular oscillations in amplitude [2,3] (Fig. 1). Both the amplitudes and times at which the wave first ceases to damp and grow (labeled "arrest" and "saturation") are power-law functions of the difference $(A_0 - A_0^\star)$ [3], thus casting the process into a *new* universality class of dynamic critical phenomena.

For a collisionless plasma the distribution function (DF) is usually not Gaussian, and because of the long-range character of the Coulomb force, these systems are outside the domain of equilibrium thermodynamics. Unlike the theory of critical phenomena in thermodynamics, where only *spatial* correlations are considered through the order parameter $\phi(\mathbf{x})$ and the partition function $Z = \int \mathcal{D}\phi(\mathbf{x}) e^{-H[\phi]}$ [4], threshold physics in collisionless systems involves correlations in *velocity* space [5]. Therefore these correlations, due to resonant energy exchange between particles and waves, are a new paradigm for critical phenomena potentially applicable in a vast class of systems, e.g., coupled phase oscillators which show Landau damping or equivalent Josephson-junction arrays [6].

Crawford's pioneering analysis [7] reveals the striking difference between thermodynamic and plasma situations due to this physics: the resonance between particles and waves at the phase velocity $v_\phi = \omega_{\rm pe}/k$ turns the thermodynamic exponent $\beta = 1/2$ [4] into the "trapping scaling" exponent $\beta = 2$, which describes saturation of the weak bump-on-tail [8] and gravitational instabilities [5]. In the frameworks of linear and quasilinear theory [9], arrest of the linear damping of plasma waves (as well as saturation of the growth) might be explained in terms of flattening of the DF at v_ϕ, thus bringing the damping (growth) rate $\gamma_{\rm L} \sim (\partial f/\partial v)_{v=v_\phi}$ of a kinetic instability to zero.

Trapping of electrons in a monochromatic wave's electric potential is often suggested as a nonlinear mechanism to stop the initial exponential damping phase and to saturate the wave's growth [10,11]. Trapping and its associated Bernstein-Greene-Kruskal (BGK) modes [12] also imply a certain shape of the DF plus trapped and untrapped orbits in velocity phase space. However, it is controversial whether trapping is relevant to the damping threshold. For instance, one analysis [13] assumes ergodicity of trapped particles in a single-wave potential and predicts the threshold initial electric field amplitude E_0 through the critical ratio $q_c = |\gamma_{\rm L}|/\omega_{\rm tr} \approx 0.06$ of the absolute Landau damping rate $|\gamma_{\rm L}|$ to the trapping frequency $\omega_{\rm tr} = (kE_0 e/m_e)^{1/2}$. In contrast, full Vlasov-Poisson (VP) simulations for a Maxwellian plasma yield $q_c \approx 0.85$ from the asymptotic evolution [14] and $q_c \approx 1.0$ from the initial evolution [3], with constants of proportionality slightly different from unity for other thermal plasmas [3].

Other conflicting evidence exists on the role of trapping. Consider the critical exponents $\tau_{\rm min}$, $\beta_{\rm min}$, $\tau_{\rm sat}$, and

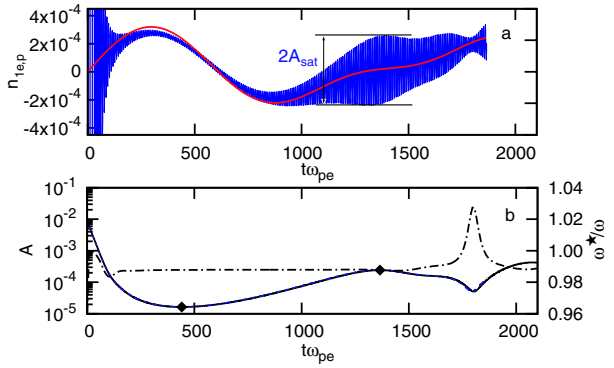

FIG. 1 (color online). (a) Two-component electron (blue or gray shading) and ion (red or light gray line) $m = 1$ field evolution for $m_p/m_e = 1836$. (b) Electric field envelope A_e for mode $m = 1$ for immobile ions (black solid line) and mobile ions $A_{(e+p)}$ (blue or gray dashed line), and the ratio of the simulated frequency to its analytic prediction, $\omega^\star/\omega$ (right axis, black dash-dotted line). Diamonds mark the arrest time $t_{\rm min}$ and saturation time $t_{\rm sat}$.

$\beta_{\rm sat}$ for the power-law functions of $(A_0 - A_0^\star)$ obeyed by, respectively, the time $t_{\rm min}$ and amplitude $A_{\rm min}$ at which the initial damping phase finishes, as well as the time $t_{\rm sat}$ and amplitude $A_{\rm sat}$ at which the first exponential growth phase saturates [3]: e.g., $t_{\rm min} \propto (A_0 - A_0^\star)^{-\tau_{\rm min}}$ and $A_{\rm min} \propto (A_0 - A_0^\star)^{\beta_{\rm min}}$. First, the temporal exponents $\tau_{\rm min} = 0.901 \pm 0.008$ and $\tau_{\rm sat} = 1.039 \pm 0.011$ are measurably different from each other and the value 0.5 expected from the definition of $\omega_{\rm tr}$. Second, the field exponents $\beta_{\rm sat} = 1.88 \pm 0.07$ and $\beta_{\rm min} = 2.72 \pm 0.09$ are remarkably different from each other and the value $\beta_{\rm tr} = 1$ expected for trapping [11]. These points argue against trapping causing either of the arrest and saturation phenomena. Third, calculations with $A_0 \gg A_0^\star$ lead to τ and β exponents closer to 0.6 and 1.3, respectively, and the oscillation spectrum has clear peaks near $\omega_{\rm tr}$, suggesting that trapping plays a role well above threshold [3].

In this Letter we first simulate one-dimensional (1D) VP two-component plasma with initially Maxwellian distributions for electrons and ions and demonstrate that ion mobility does not affect the threshold phenomenon for Langmuir wave damping seen in VP simulations without ions. Then, using one-component electron VP simulations, we demonstrate that the DF phase portrait when the wave first ceases to damp is much simpler than a BGK equilibrium [12] and shows no evidence for trapping. Instead, we demonstrate that the initial DF resonantly evolves a positive slope in velocity space that stops the initial Landau damping and supports the subsequent exponential growth. We also demonstrate that the DFs are different at the arrest and saturation times and are not consistent with trapping.

To clarify the importance of ion mobility, we employ first the two-component 1D VP model, normalizing to electron quantities:

$$\partial f_a/\partial t + v\partial f_a/\partial x - \mu_a E \partial f_a/\partial v = 0, \qquad (1)$$

$$\partial E/\partial x = \int_{-\infty}^{+\infty} (f_p - f_e) dv. \qquad (2)$$

Here $a = e, p$, m_a is the particle mass, f_a is the component's DF, $\mu_e = 1$, $\mu_p = -m_e/m_p$, and $E(x, t)$ is the electric field. The boundary conditions are assumed to be periodic. The initial electron distribution is

$$f_e(x, v, 0) = 1/\sqrt{2\pi} v_{{\rm th}e} \exp(-v^2/2v_{{\rm th}e}^2)[1 + A_0 \cos(k_m x)],$$

where $v_{{\rm th}e}$ is the Maxwellian thermal speed for electrons, A_0 the initial electric amplitude, $k_m = 2\pi m/L$ is the wave number of the mode m, and L is the length of the system. The ions are initially uniform and Maxwellian distributed in velocity space with $T_p = T_e$.

The simulations use $m = 1$, $v_{{\rm th}e} = 0.4$, Debye length $\lambda_{{\rm D}e} \approx 0.31$, and $L = 2\pi \approx 20.18\lambda_{{\rm D}e}$. They have $N_x = 256$ cells in the x direction both for electrons and ions, and $N_{ve} = 20\,000$ and $N_{vi} = 2000$ cells in speed for electrons and ions, respectively, within the domains $[-10v_{{\rm th}a}, 10v_{{\rm th}a}]$. The Cheng-Knorr method [15] was used to solve Eqs. (1) and (2) with double precision. System invariants $I_{3a} = \int f_a^3 dx dv$ are conserved better than $\Delta I_{3e}/I_{3e}(0) < 10^{-6}$ for electrons, and $\Delta I_{3p}/I_{3p}(0) < 10^{-9}$ for ions.

Figure 1 shows the evolution of the mode $m = 1$ for initial amplitude $A_0 = 0.012$, $A_0^\star = (8.51 \pm 0.06) \times 10^{-3}$, and $m_p/m_e = 1836$, $n_{1e,p}(t) = \int_{-\infty}^{\infty} f_{1e,p}(v, t) dv$. This type of evolution is observed experimentally [16]. The existence of significant ion motion in Fig. 1(a) seems, at first glance, to suggest that the evolution is seriously affected by ion mobility. However, the envelope field amplitude of the electron oscillations is identical to within 1% for mobile and immobile ions [Fig. 1(b)]. Quantitatively, the initial damping phase in Fig. 1 stops at time $t_{\rm min} \approx 441\omega_{\rm pe}^{-1}$ and amplitude $A_{\rm min} \approx 1.64 \times 10^{-5}$, and is then followed by almost exponential growth which saturates at $t_{\rm sat} \approx 1365\omega_{\rm pe}^{-1}$ and $A_{\rm sat} \approx 2.42 \times 10^{-4}$. These quantities are identical to those calculated in the electron VP simulations of Ref. [3], where $m = 4$ was assumed for the perturbation and $v_{\rm th} = 0.1$ for the electron thermal speed. This is expected because $k\lambda_{{\rm D}e}$, the wave frequency ω, and $\gamma_{\rm L}$ are the same for the two simulations.

Analytic theory predicts that $\omega \approx 1.2851\omega_{\rm pe}$, but the simulated value $\omega^\star = 1.2705 \pm 9 \times 10^{-4}$ is slightly shifted from ω due to the large value of A_0 and varies slightly with time [Fig. 1(b)]. The linear damping rate is $\gamma_{\rm L} \approx -0.0661\omega_{\rm pe}$. For smaller $A_0 = 10^{-5}$ both $\omega^\star$ and $\gamma_{\rm L}^\star$ match the standard Landau theory [1] very well (not shown), with $\{|(\omega^\star - \omega)/\omega|, |(\gamma_{\rm L}^\star - \gamma_{\rm L})/\gamma_{\rm L}|\} < 2 \times 10^{-4}$.

These two-component VP results demonstrate that the threshold phenomenon for Langmuir wave damping is robust against ion effects: $\max[|A_e(t) - A_{(e+p)}(t)|/A_e(t)] < 10^{-2}$ on the interval $0 \le t \le t_{\rm sat}$. Accordingly, one-component simulations, with ions acting as a neutralizing background, are used below.

The DF near the phase velocity $v_\phi = \omega^\star/k_1 \approx 1.271$ at $t = t_{\rm min}$ and $t = t_{\rm sat}$ is shown in Fig. 2 and reveals drastic discrepancies between the evolution which ends with arrest of damping at $t = t_{\rm min}$ and the subsequent evolution until the growth saturates at $t = t_{\rm sat}$. At $t = t_{\rm min}$ the phase space portrait reveals no signs of particle trapping—only filamentation due to phase mixing (Fig. 2, upper panel). Moreover, instead of a stationary state this distribution supports approximately linear (meaning exponential) growth on the interval $t_{\rm min} < t < t_{\rm sat}$, as Fig. 1(b) shows. Crucially, the DF at $t_{\rm sat}$ does not consist of the closed orbits (or whorls in velocity-position space) expected for trapping. Instead, the orbits are still open, although they clearly indicate progress towards trapping. Trapping is therefore responsible for neither the arrest of damping nor the saturation of the growth phase.

In the linear theory developed by Landau [1] growth is due to a positive slope at the phase velocity of the wave, $|v| = v_\phi$, in the DF averaged over the x coordinate, $f_0(v, t) = (1/L) \int_0^L f(x, v, t) dx$. Figure 3 shows $f_0(v, t = t_{\rm min})$. Instead of the flattening of f_0 near the resonant

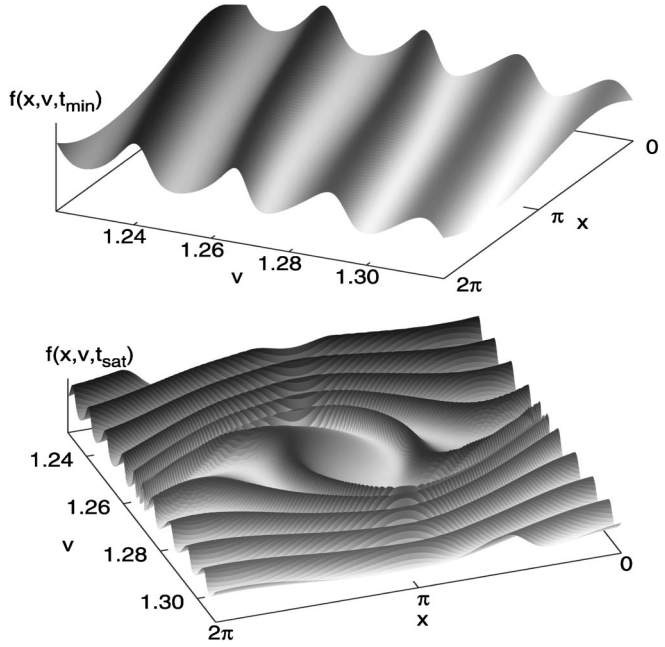

FIG. 2. DF when (upper panel) $t = t_{\min}$ and damping stops and (lower panel) $t = t_{\text{sat}}$ and growth saturates.

velocities $v = \pm v_\phi$ predicted by quasilinear theory [9], $f_0(v, t_{\min})$ acquires a *positive* slope in a small vicinity of v_ϕ, and therefore can support (approximately) linear growth after $t = t_{\min}$ as Fig. 1(b) shows.

Contrary to the situation near $t = t_{\min}$ when damping ceases and the physics looks quite smooth and regular, f_0 becomes quite irregular near the time $t = t_{\text{sat}}$ when growth saturates (Figs. 2 and 4). In particular, the lower panel of Fig. 2 is strongly reminiscent of trapping, although strictly closed trajectories do not appear for this A_0. Also, while on average the slope of $f_0(v, t_{\text{sat}})$ at $v = \pm v_\phi$ seems to have decreased compared with time $t = t_{\min}$ [Fig. 4(a)], it varies irregularly in the neighborhood of $\pm v_\phi$ and therefore may support excitation of oscillations with a wide range of phase speeds.

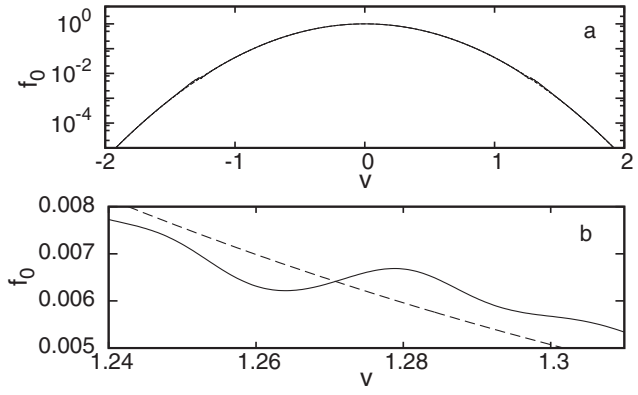

FIG. 3. $f_0(v, t_{\min})$ (solid lines) and $f_0(v, 0)$ (dashed line) on semilogarithmic (upper panel) and linear (lower panel) scales for two velocity intervals: (a) $-2.0 \le v \le 2.0$ and (b) $1.24 \le v \le 1.32$.

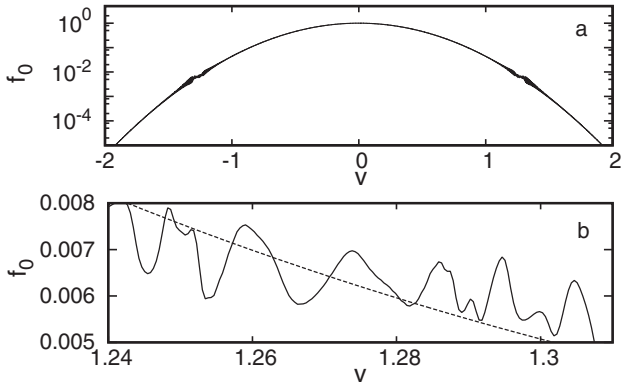

FIG. 4. $f_0(v, t_{\text{sat}})$ (solid lines) and $f_0(v, 0)$ (dashed line) on semilogarithmic (upper panel) and linear (lower panel) scales for two velocity intervals: (a) $-2.0 \le v \le 2.0$ and (b) $1.24 \le v \le 1.32$.

Figure 5 shows the evolution of, and power transfers between, the average DF $f_0(v, t)$ and the DF components $f_1(v, t)$ and $f_2(v, t)$ at k_1 and k_2, respectively, with $|f_m(v, t)| = \{\text{Re}^2[f_m(v, t)] + \text{Im}^2[f_m(v, t)]\}^{1/2}$. It shows that the dynamical picture can be divided into regions with distinct characteristics that identify the processes causing the evolution. Figure 5 shows that the turbulent processes responsible for the (relative) flattening of f_0 in the resonant area near v_ϕ start only *after* t_{sat}, when spatial Fourier components E_m other than $m = 1$ become comparable to E_1 (not shown here).

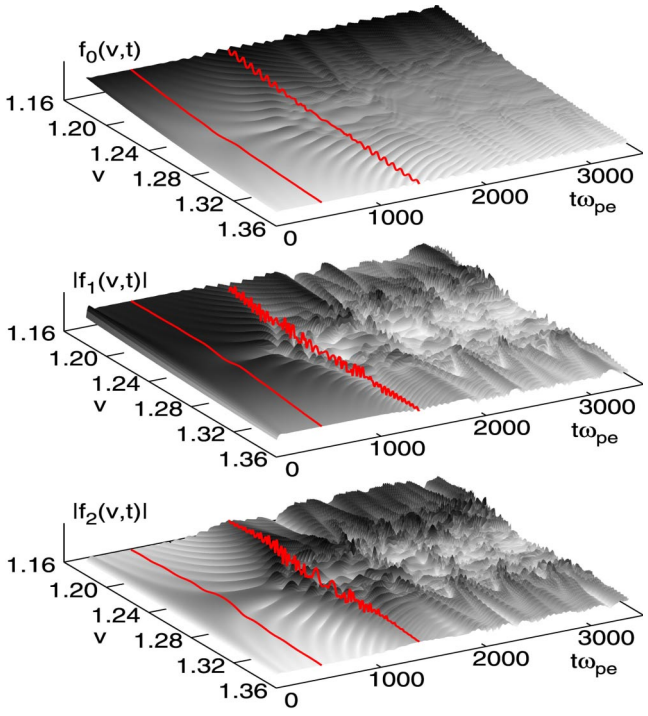

FIG. 5 (color online). Evolution of DF components f_0 (top panel), f_1 (middle panel) and f_2 (bottom panel). Red or gray lines are DF components at the moments $t_{\min}$ and t_{sat}, e.g., $|f_1(v, t_{\text{sat}})|$.

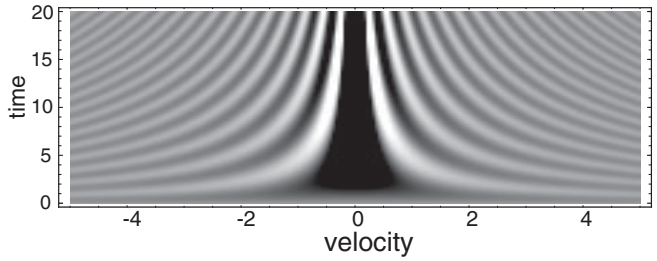

FIG. 6. Power transfer $P(v)$ for a damping wave with $\gamma = 0.03$ in Eq. (15) of Ref. [17].

The ripples of f_0, $|f_1|$, and $|f_2|$ in time and velocity appear to be ''fingerprints'' of ballistic change of initial perturbation and power transfer between the field and particles. The latter claim is justified by Fig. 6, which illustrates the power transfer rate for a wave growing/damping linearly by resonant wave-particle interactions given by Eq. (15) in Ref. [17]. Together with Fig. 7, which shows the evolution of $\delta f_0(v, t) = [f_0(v, t) - f_0(v, 0)]/f_0(v, 0)$ on the interval $0 \le t \le t_{\min}$, Figs. 5 and 6 clearly demonstrate that the physical process responsible for arrest of linear damping is the resonant power transfer between the wave and the $m = 0$ and higher order components of the DF.

An insight into the striking difference between the critical exponents $\beta_{\min}$ and β_{sat} comes from critical phenomena theory: critical exponents depend on the properties of correlations for a specific system (e.g., on its dimensionality) and/or a universality class (e.g., Ising, percolation, surface growth, etc.) [4]. The DFs in full phase space (position and velocity) are different at times $t_{\min}$ and t_{sat} (see Figs. 2–5 and 7), so the critical exponents might be different. This difference is contrary to the idea that trapping explains both the arrest and saturation phases, which should result in the *same* exponents. Some plausibility for velocity-space structures having this effect follows from 1D VP self-gravitating calculations: varying the resolution in v seriously affected estimates of the trapping scaling exponent $\beta = 2$ [5].

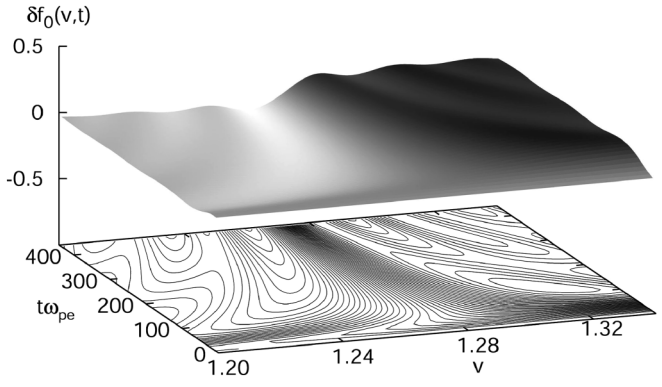

FIG. 7. Evolution of $\delta f_0(v, t)$ until the moment $t = t_{\min}$.

In summary, we studied the VP model for initial Langmuir wave amplitudes slightly above the threshold that separates damping and nondamping evolution. Electron-ion simulations show that ion mobility does not modify the threshold found for Langmuir damping in electron-only simulations. Phase space diagnostics show no signs of trapping or the DF flattening near $t = t_{\min}$—instead the combined effects of ballistic evolution of perturbations and resonant power transfer at $|v| \approx v_\phi$ are responsible for arrest of the linear (Landau) damping then. Since the spatially averaged DF is not flat at $t_{\min}$ but instead has a positive slope near the resonant velocity v_ϕ, this state is not stationary but instead leads to (linear) growth which is saturated at $t = t_{\text{sat}}$. The saturation time t_{sat} marks the boundary between the regular and stochastic evolution of the wave electric field, again with no evidence for trapping saturating the growth phase.

[1] L. D. Landau, J. Phys. (Moscow) **10**, 25 (1946).
[2] R. Sugihara and T. Kamimura, J. Phys. Soc. Jpn. **33**, 206 (1972); J. Canosa and J. Gazdag, Phys. Fluids **17**, 2030 (1974).
[3] A. V. Ivanov, I. H. Cairns, and P. A. Robinson, Phys. Plasmas **11**, 4649 (2004).
[4] N. Goldenfeld, *Lectures on Phase Transitions and the Renormalization Group* (Perseus Books, Reading, MA, 1992); H. Hinrichsen, Adv. Phys. **49**, 815 (2000).
[5] A. V. Ivanov, Astrophys. J. **550**, 622 (2001); A. V. Ivanov, S. V. Vladimirov, and P. A. Robinson, Phys. Rev. E **71**, 056406 (2005).
[6] S. H. Strogatz, R. E. Mirollo, and P. C. Matthews, Phys. Rev. Lett. **68**, 2730 (1992); S. H. Strogatz, Physica (Amsterdam) **143D**, 1 (2000); J. A. Acebrón *et al.*, Rev. Mod. Phys. **77**, 137 (2005); V. Latora, A. Rapisarda, and S. Ruffo, Physica (Amsterdam) **131D**, 38 (1999).
[7] J. D. Crawford, Phys. Rev. Lett. **73**, 656 (1994).
[8] E. Frieman, S. Bodner, and P. Rutherford, Phys. Fluids **6**, 1298 (1963).
[9] W. E. Drummond and D. Pines, Nucl. Fusion Suppl. **3**, 1049 (1962); A. A. Vedenov, E. D. Velikhov, and R. Z. Sagdeev, Nucl. Fusion Suppl. **2**, 465 (1962).
[10] T. O'Neil, Phys. Fluids **8**, 2255 (1965); F. Valentini, V. Carbone, P. Veltri, and A. Mangeney, Phys. Rev. E **71**, 017402 (2005).
[11] C. Lancellotti and J. J. Dorning, Phys. Rev. Lett. **81**, 5137 (1998); Phys. Rev. E **68**, 026406 (2003).
[12] I. B. Bernstein, J. M. Greene, and M. D. Kruskal, Phys. Rev. **108**, 546 (1957); M. Buchanan and J. Dorning, Phys. Rev. E **50**, 1465 (1994).
[13] M. C. Firpo and Y. Elskens, Phys. Rev. Lett. **84**, 3318 (2000).
[14] M. Brunetti, F. Califano, and F. Pegoraro, Phys. Rev. E **62**, 4109 (2000).
[15] C. Z. Cheng and G. Knorr, J. Comput. Phys. **22**, 330 (1976).
[16] J. R. Danielson, F. Anderegg, and C. F. Driscoll, Phys. Rev. Lett. **92**, 245003 (2004).
[17] W. E. Drummond, Phys. Plasmas **11**, 552 (2004).